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OBTAIN ALGORITHM
FOR THE SUPPLY CHAIN NETWORK EQUILIBRIUM PROBLEM

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Abstract: An origin-based solution algorithm, named OBTAIN (origin-based traffic assignment for infrastructure networks), was recently developed for solving the traffic assignment problems (Bar-Gera and Boyce, 2003). This algorithm defines the solution variables in an intermediate way between links and routes, and has been proven to be superior to the link-based Frank-Wolfe method in both computational time for and accurate level of the solution for a large network. To address its usefulness, the OBTAIN algorithm is elaborated in a tutorial manner, especially in its two major steps for updating the restricting subnetworks and origin-based approach proportions, and then applied to a small supply chain network equilibrium problem. The obtained results comply with the sufficient conditions for restricted user equilibrium. Future research includes a real application of large networks and a comparison of the OBTAIN algorithm with the route-based gradient projection (GP) algorithm in terms of computational efficiency.

Key Words: OBTAIN method, Frank-Wolfe method, traffic assignment, supply chain network equilibrium problem.

1. INTRODUCTION

Traffic assignment, which is governed by Wardrop principles, is embedded in a variety of transportation network models. Many heuristics such as capacity restraint and incremental methods were developed for approximately obtaining the solution. The drawback of those heuristics is that the obtained solution is hardly to be accurate. To improve the level of accuracy of the solution and hence comply with the associated optimality/equilibrium conditions, the Frank-Wolfe (FW) algorithm (Frank et al., 1956) was introduced by LeBlanc et al. (1975) to solve the traffic assignment optimization problem. The advantage of the FW algorithm is that it only requires mild computer memory, which accounts for a large proportion of cost for purchasing a computer in the past. However, with the advancement of today's technology, the amount of computer memory requirement is not a big concern in terms of cost. On the other hand, the disadvantages of the FW method are twofold: (1) slow convergence rate and (2) providing no route information. The slow convergence rate is more pronounced when the network size is large, which is typically seen in real-time and dynamic networks. Route information is important in providing guidance for travelers, and in dispatching emergency vehicles for evacuation in disastrous areas. As a result, new algorithms that converge faster and in the meantime provide route information are desired. For this purpose, a set of route-based solution algorithms such as disaggregated simplicial algorithm (Larsson et al., 1992) and the gradient projection (GP) method (Jayakrishnan et al., 1994) has gained much attention from researchers in transportation community. The interested reader may also refer to Florian and Hearn (1993) and Patriksson (1994) for more route-based solution algorithms. It is noted that the GP method has become most popular as it has already been proven superior to the FW method in terms of computational efficiency both for the dynamic traffic assignment problem (Chen et al., 1999) and for static traffic assignment problem (Tatineni et al., 1998).

Recently, a new category of origin-based solution algorithms, called OBTAIN (origin-based traffic assignment for infrastructure networks), was also made available in the literature (Bar-Gera, 1999, 2002; Bar-Gera and Boyce, 2003). The OBTAIN algorithm is between the link-based and route-based solution algorithms and has been demonstrated to outperform the FW method with respect to computational performance for several large networks including Chicago Network. Though the OBTAIN algorithm has been elaborately studied from both theoretical and practical points of view, it remains to have many nice features to be further exploited. In this paper, the OBTAIN algorithm is elaborated in a tutorial manner, especially in its two major steps for updating the restricting subnetworks and origin-based approach proportions, and then applied to a small supply chain network equilibrium (SCNE) problem.

In the following, we begin with the traffic assignment formulation and its optimality conditions in Section 2. The OBTAIN method and its algorithmic steps are described in Section 3. A numerical example of the SCNE problem is then demonstrated in Section 4. Finally few remarks are given in Section 5.

2. TRAFFIC ASSIGNMENT PROBLEM

The main theme of this paper is to elaborate the OBTAIN algorithm and then explore its application to a SCNE problem. As implied by the name of the OBTAIN, the algorithm is closely associated with traffic assignment and can therefore be decomposed by origins. Therefore, for simplicity, the model formulation and the related discussion hereafter will be restricted to the traffic assignment problem with a single origin $\bar{r}$. The restricting network associated with origin $\bar{r}$ is denoted by $A^{\bar{r}}$. Note that the results can be naturally extended to the situation of multiple origins without any difficulty.

2.1 Model Formulation

Traffic assignment approximates route choice behavior of travelers in a network in which Wardrop's two principles are commonly used. Wardrop's first (or user optimal) principle states that for each origin-destination (O-D) pair, every traveler searches for the shortest route whereas Wardrop's second (or system optimal) principle requires that the total network travel cost is minimum. The two Wardrop principles can be addressed in a similar manner, with at most minor modifications. For the purpose of illustration, we choose Wardrop first principle for use throughout the paper. Assuming that link travel cost c_a is the function of flows x_a on link *a*, i.e., $c_a(x_a)$, and hence separable, many mathematical models such as optimization, nonlinear complementarity, variational inequality and fixed point formulations are available for formulating the traffic assignment problem. Here we use optimization formulation.

$$\min_{\mathbf{h}\in\Omega} \ z(\mathbf{x}(\mathbf{h})) = \sum_{a\in A^{\bar{r}}} \int_0^{x_a(\mathbf{h})} c_a(\omega)d\omega \tag{1}$$

where feasible region Ω is delineated by the following constraints.

Flow conservation constraint:

$$\sum_{p\in P_j^{\bar{r}}\left[A^{\bar{r}}\right]} h_{pj}^{\bar{r}} = q_j \qquad \forall j \in N \tag{2}$$

Nonnegativity constraint:

$$h_p^{\bar{r}s} \geq 0 \qquad \forall s \in S(\bar{r}), p \tag{3}$$

Objective (1) is to minimize the sum of the integral of the link travel cost $c_a(x_a)$ over all links in the restricting network associated with origin $\bar{r}$. Constraint (2) conserves flow q_j for each node j in terms of origin-based route flows $\{h_{pj}^{r}\}$ in the restricting network. Constraint (3) requires route flows $\{h_p^{\bar{r}s}\}$ be nonnegative.

2.2 Optimality Conditions

If we let origin-based approach proportions $\{\alpha_a\}$ be the decision variables of the traffic assignment problem (1), the derivative of objective function z with respect to approach proportion α_a can be written, using chain rule, as follows.

$$\frac{\partial z}{\partial \alpha_a} = \sum_{a'\in A^r} \frac{\partial z}{\partial x_{a'}} \frac{\partial x_{a'}}{\partial \alpha_a} \tag{4}$$

The first term on the right hand side simply yields $\frac{\partial z}{\partial x_{a'}} = c_{a'}$. The second term on the right hand side need consider four cases: a=a'; $o(a_h) = o(a'_h), a' \neq a$; $o(a'_h) < o(a_h)$; and $o(a_h) < o(a'_h)$, where $o(\bullet)$ represents the topological order of the node denoted by the argument $(\bullet)$, and a_h the head of link a. For each node j, the feasibility constraint is as follows:

$$\sum_{a\in A^r, a_h = j} \alpha_a^r = 1 \qquad \forall j \in N \setminus \{r\} \tag{5}$$

If we choose one approach $b_j \in A^r$, $(b_j)_h = j$ as *basic* approach, and denote all other approaches (if there are any) as the *non-basic* approaches, $NB_j = \{a \in A^r, a_h = j, a \neq b_j\}$; $NB = \{\bigcup_{j\in N} NB_j\}$. The above feasibility constraint can be alternatively written as

$$\alpha_{b_j} = 1 - \sum_{a\in NB_j} \alpha_a^r \qquad \forall j \in N \setminus \{r\} \tag{6}$$

Consider the aforementioned four cases and denote the head node by $j = a_h$, its basic approach $b = b_j = b_{(a_h)}$, the "constrained" derivative of objective function z with respect to the approach proportion of non-basic approach α_a^{NB} can be written as follows.

$$
\begin{aligned}
&\frac{\partial z}{\partial^c \alpha_a^{NB}} = \sum_{a' \in A^r} \frac{\partial z}{\partial x_{a'}} \frac{\partial x_{a'}}{\partial \alpha_a} \\
&= c_a \cdot q_j - c_b \cdot q_j + \sum_{a' \in A^r; o(a_h^{'}) < o(j)} c_{a'} \cdot \alpha_{a'} \cdot q_j \cdot \left(\chi_{a_h^{'} \to a_t} - \chi_{a_h^{'} \to b_t} \right) \\
&= q_j \cdot \left(c_a + \sum_{a' \in A^r; o(a_h^{'}) < o(j)} c_{a'} \cdot \alpha_{a'} \cdot \chi_{a_h^{'} \to a_t} \right) \\
&\quad - q_j \cdot \left(c_b + \sum_{a' \in A^r; o(a_h^{'}) < o(j)} c_{a'} \cdot \alpha_{a'} \cdot \chi_{a_h^{'} \to b_t} \right) \\
&= q_j \cdot \left(c_a + \sigma_{a_t} \right) - q_j \cdot \left(c_b + \sigma_{b_t} \right) = q_j \cdot \left(\mu_a - \mu_b \right) \quad \forall r, a \in A(r)
\end{aligned}
\tag{7}
$$

where μ_a (or μ_b), σ_{a_t} (or σ_{b_t}), and $\chi_{a_h^{'} \to b_t}$ are respectively the average cost for approach a (or b), the average cost to node a_t (or b_t), and proportion of flow from head node of link a', i.e., $a_h^{'}$, to the tail node of link b, i.e., b_t. Subsequently, the first order necessary conditions for optimality are

$$\mu_a > \mu_b \quad \forall j \in N \setminus \{r\}, a \in NB_j \tag{8}$$

$$\alpha_a \cdot q_j \cdot \left(\mu_a - \mu_{b_j} \right) = 0 \quad \forall j \in N \setminus \{r\}, a \in NB_j \tag{9}$$

Note that the above conditions do not satisfy the user equilibrium conditions. First order conditions are not sufficient for optimality only if the function z is not convex. By omitting the node flow from (9), the sufficient conditions for restricted user equilibrium are obtained (Lemma 3, Bar-Gera, 2002).

$$\mu_a > \mu_{b_j} \quad \forall j \in N \setminus \{r\}, a \in NB_j \tag{10}$$

$$\alpha_a \cdot \left(\mu_a - \mu_{b_j} \right) = 0 \quad \forall j \in N \setminus \{r\}, a \in NB_j \tag{11}$$

3. ORIGIN-BASED SOLUTION ALGORITHM

In solving the traffic assignment problem, the algorithm starts with trees of minimum cost routes as restricting subnetworks, leading to an all-or-nothing assignment. Then, the algorithm considers all origins in a sequential order. For each origin the restricting subnetwork is updated, and the origin-based approach proportions are adjusted within the given restricting subnetwork. Therefore, the OBTAIN method contains two major steps for each origin r: (1) initialize/update restricting subnetworks, (2) initialize/update origin-based approach proportions.

3.1 Restricting Subnetworks

A key point in the algorithm is that for every origin an *a-cyclic* restricting subnetwork, A^r, is chosen such that, for origin r, approach proportions of links that are not included in A^r are restricted to 0. Using the equation for route proportions, it can be seen that under these restrictions, for every origin only routes that are limited to the links in its restricted subnetwork can be used.

Since A^r is a-cyclic, meaning that it does not contain a directed cycle of links, any cyclic route must contain at least one link that does not belong to A^r, and hence the flow along any cyclic route must be zero. It is important to note that the restriction to a-cyclic subnetworks does exclude many solutions that do not use cyclic routes, which are usually considered legitimate. Bar-Gera (2002; Lemma 3), among others, has shown that there is always a *user equilibrium* solution that is a-cyclic by origin. Therefore, this restriction does not prevent the algorithm from converging to the true equilibrium solution.

The restriction to solutions that are a-cyclic by origin has several important advantages. First, the simple route flow interpretation presented above is, in fact, only valid for solutions that are a-cyclic by origin. Second, a-cyclic subnetworks allow a definition of a topological order of the nodes, which is an origin-specific ordering of the nodes, such that every link in the restricting subnetwork goes from a node of lower topological order to a node of higher topological order. Most computations in the proposed algorithm are done in a single pass over the nodes, either in ascending or descending topological order. The time required by such computations is a linear function of the number of links in the network, i.e., CPU_Time $\propto |N|$. This computational efficiency is the main reason for restricting subnetworks to a-cyclic solutions.

To update a restricting subnetwork, unused links are removed; u_i the maximum cost from the origin to node i within the restricting subnetwork is computed for all nodes, and all links [i,j] in which $u_i < u_j$ are added to the restricting subnetwork. Once a new restricting subnetwork is found, several computationally intensive steps are needed, including reorganization of the data structure. However, restricting subnetworks tend to stabilize fairly quickly.

3.2 Origin-Based Approach Proportions

The main solution variables in this algorithm are *origin-based approach proportions*, α_a^r, for every origin r and every link a, such that for every origin r and node j the sum of the origin-based approach proportions over all links ending at node j is equal to one, as already shown in (5). Using origin-based approach proportions, *route proportions* are determined as the product of the approach proportions of all the links along the route, that is $\gamma_p^{rs} = \prod_{a \subseteq p} \alpha_a^r$.

Route flows are determined as the product of O-D flow and route proportion, that is $h_p^{rs} = q^r \cdot \gamma_p^{rs}, \forall p \in P^{rs}$. Bar-Gera (2002; eq. 14) has shown that if link a goes from node i to node j, i.e., $a_t = i, a_h = j$, and if the total flow from origin r to node j is q_j^r, then the total flow from origin r that arrives at node j through link a is $x_a^r = \alpha_a^r \cdot q_j^r$; in this respect, α_a^r is indeed the proportion of flow on approach a to node j for origin r, as implied from the name of these variables.

The representation of the solution by origin-based approach proportions allows storing a complete description of the *route flows* very efficiently. The efficiency of the representation is further enhanced because of the fact that at most nodes one link receives an approach proportion value of one, while the value of all other links ending at the same node is zero. The availability of route flows can be useful for solution analysis. It is also useful in searching for the *equilibrium solution*, which is a major difference from many alternative solution procedures, including the FW algorithm, which stores only total link flows during the iterative process.

Origin-based approach proportions are updated while keeping the restricting subnetworks fixed. To update origin-based approach proportions within a given restricting subnetwork, a search direction based on shifting flow from high-cost alternatives to low-cost alternatives, that have been treated in (7), is used.

$$\frac{\partial z}{\partial^c \alpha_a^{NB}} = q_j \bullet (\mu_a - \mu_b) \quad \forall a, j \in N \setminus \{r\} \tag{12}$$

where the average cost for approach *a* (or *b*), μ_a (or μ_b), and the average cost to node $j = a_h = b_h$, i.e., σ_j, can be computed recursively by

$$\sigma^r(\boldsymbol{\alpha}, \mathbf{c}) = \sigma_0(\boldsymbol{\alpha}, \mathbf{c}) = 0$$

$$\sigma_j(\boldsymbol{\alpha}, \mathbf{c}) = \sum_{a' \in A^r; a_h = j} \alpha_a \bullet \mu_a(\boldsymbol{\alpha}, \mathbf{c}) \quad \forall j \in N \setminus \{r\} \tag{13}$$

$$\mu_a(\boldsymbol{\alpha}, \mathbf{c}) = c_a + \sigma_{a_t}(\boldsymbol{\alpha}, \mathbf{c}) = c_a + \sum_{a' \in A^r; a_h^{'} = a_t} \alpha_{a'} \bullet \mu_{a'}(\boldsymbol{\alpha}, \mathbf{c})$$

In addition to current costs, estimates of the "constrained" cost derivatives are used to improve the search direction in a quasi-Newton fashion.

$$\frac{\partial^2 z}{\partial^c \alpha_a^{NB2}} = \sum_{a' \in A^r} \left[\frac{\partial^2 z}{\partial x_a^{\ 2}} \left(\frac{\partial x_{a'}}{\partial^c \alpha_a} \right)^2 + \frac{\partial z}{\partial x_{a'}} \left(\frac{\partial^2 x_{a'}}{\partial^c \alpha_a^{\ 2}} \right) \right] \tag{14}$$

Since the diagonal second order derivatives of the flow on any link *a'* with respect to any other approach proportion *a* is always zero, i.e., $\frac{\partial^2 x_{a'}}{\partial^c \alpha_a^{\ 2}} = 0$, the above formula reduces to

$$\frac{\partial^2 z}{\partial^c \alpha_a^{NB2}} = \sum_{a' \in A^r} \frac{\partial^2 z}{\partial x_a^{\ 2}} \left(\frac{\partial x_{a'}}{\partial^c \alpha_a} \right)^2 \tag{15}$$

The first term on the right hand side yields $\frac{\partial^2 z}{\partial x_a^{\ 2}} = c_a^{'}$. The second term on the right hand side again need consider four cases: *a=a'*; $o(a_h) = o(a_h^{'}), a' \neq a$; $o(a_h^{'}) < o(a_h)$; and $o(a_h) < o(a_h^{'})$, where $o(\bullet)$ represents the topological order of the node denoted by the argument $(\bullet)$. Subsequently, the "constrained" cost derivatives can be expressed as follows:

$$\frac{\partial^2 z}{\partial^c {\alpha_a^{NB}}^2} = c_a^{'} \bullet q_j^2 + c_b^{'} \bullet q_j^2 + \sum_{a' \in A^r; o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet q_j^2 \bullet \left(\chi_{a_h^{'} \to a_t} - \chi_{a_h^{'} \to b_t}\right)^2$$
$$= c_a^{'} \bullet q_j^2 + c_b^{'} \bullet q_j^2 + \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet q_j^2 \bullet \left(\chi_{a_h^{'} \to a_t} - \chi_{a_h^{'} \to b_t}\right)^2 \qquad (16)$$

Since $\left(\chi_{a_h^{'} \to a_t} - \chi_{a_h^{'} \to b_t}\right)^2$ can be further written out, the above expression can be estimated as follows:

$$\frac{\partial^2 z}{\partial^c {\alpha_a^{NB}}^2} \approx c_a^{'} \bullet q_j^2 + \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet q_j^2 \bullet \chi_{a_h^{'} \to a_t}^2$$
$$+ c_b^{'} \bullet q_j^2 + \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet q_j^2 \bullet \chi_{a_h^{'} \to b_t}^2$$
$$-2 \bullet \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet q_j^2 \bullet \chi_{a_h^{'} \to b_t}^2$$
$$\approx q_j^2 \bullet \begin{pmatrix} c_a^{'} + \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet \chi_{a_h^{'} \to a_t}^2 \\ + c_b^{'} + \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet \chi_{a_h^{'} \to b_t}^2 \\ -2 \bullet \sum_{a' \in A^r; o(lcn_j) < o(a_h^{'}) < o(j)} c_{a'}^{'} \bullet \alpha_{a'}^2 \bullet \chi_{a_h^{'} \to b_t}^2 \end{pmatrix}$$
$$\approx q_j^2 \bullet \left(c_a^{'} + \rho_a + c_b^{'} + \rho_b - 2 \bullet \rho_{lcn_j}\right)$$
$$\approx q_j^2 \bullet \left(\upsilon_a + \upsilon_b - 2 \bullet \rho_{lcn_j}\right) \quad \forall r, a \in A(r), j \in N \setminus \{r\} \qquad (17)$$

where lcn_j denotes the *last common node* to node *j*, and υ_a (or υ_b) and ρ_j are respectively the average cost derivative for approach *a* (or *b*) and the average cost derivative to node $j = a_h = b_h$, which can be computed recursively by

$$\rho^r(\boldsymbol{\alpha}, \mathbf{c'}) = \rho_0(\boldsymbol{\alpha}, \mathbf{c'}) = 0$$
$$\rho_j(\boldsymbol{\alpha}, \mathbf{c'}) = \sum_{a' \in A^r; a_h = j} \alpha_a^2 \bullet \upsilon_a(\boldsymbol{\alpha}, \mathbf{c'}) \quad \forall j \neq r \qquad (18)$$
$$\upsilon_a(\boldsymbol{\alpha}, \mathbf{c'}) = c_a^{'} + \rho_{a_t}(\boldsymbol{\alpha}, \mathbf{c'}) = c_a^{'} + \sum_{a' \in A^r; a_h^{'} = a_t} \alpha_{a'}^2 \bullet \upsilon_{a'}(\boldsymbol{\alpha}, \mathbf{c'})$$

The desirable amount of flow proportion (*DFP*) to be shifted between two alternative approaches *a* and *b* ($a_h = b_h = j$) is determined by a Newton type shift (the second-order search direction) as follows.

$$DFP = \frac{\partial z}{\partial^c \alpha_a^{NB}} / \frac{\partial^2 z}{\partial^c \alpha_a^{NB\,2}} = \frac{q_j \bullet (\mu_a - \mu_b)}{q_j^2 \bullet (\upsilon_a + \upsilon_b - 2 \bullet \rho_{lcn_j})} = \frac{1}{q_j} \bullet \frac{(\mu_a - \mu_b)}{(\upsilon_a + \upsilon_b - 2 \bullet \rho_{lcn_j})} \quad \forall a \in A(r), j \in N \setminus \{r\} \tag{19}$$

Since both the node flow q_j and the second denominator component $(\upsilon_a + \upsilon_b - 2 \bullet \rho_{lcn_j})$ may have a value of zero, a technique of avoiding the above equation to be ill-defined is adopted as follows.

$$\alpha_{a \to b}(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{c}') = \begin{cases} \left\{ \min\left(\alpha_a, \dfrac{x_{a \to b}(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{c}')}{q_j(\boldsymbol{\alpha})} \right) \right\} & q_j > 0 \\ \{\alpha_a\} & q_j = 0; \mu_a > \mu_b \quad \forall a, j \in N \setminus \{r\} \\ [0, \alpha_a] & q_j = 0; \mu_a = \mu_b \end{cases} \tag{20}$$

where the desirable shifted flow $x_{a \to b}(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{c}')$ is defined as

$$x_{a \to b}(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{c}') = q_j \bullet DFP_{\varepsilon_\upsilon} \quad \forall r, s, a \tag{21}$$

and $DFP_{\varepsilon_\upsilon}$ is a modification of DFP using a small positive constant ε_υ so as to overcome the problem of "zero" derivative estimate:

$$DFP_{\varepsilon_\upsilon} = \frac{1}{q_j} \bullet \frac{(\mu_a(\boldsymbol{\alpha}, \mathbf{c}) - \mu_b(\boldsymbol{\alpha}, \mathbf{c}))}{\max(\varepsilon_\upsilon, \upsilon_a(\boldsymbol{\alpha}, \mathbf{c}') + \upsilon_b(\boldsymbol{\alpha}, \mathbf{c}') - 2 \bullet \rho_{lcn_j}(\boldsymbol{\alpha}, \mathbf{c}'))} \quad \forall a, j \in N \setminus \{r\} \tag{22}$$

Note that the second and third parts of the right hand side in (20) are designed to take care of the problem caused by the zero node flow, that is $q_j = 0$.

Two different approaches are possible for determining the step size λ: (a) guarantee feasibility first then address the "optimal" step size (b) scale the second-order search direction by a step size between zero and one, and then truncated to guarantee feasibility. The conventional line search techniques, in which shifts are first truncated to guarantee feasibility and only then scaled by a step size, usually adopts the concept of the first approach. One well known method is the FW method which is easy to understand but converges rather slow due to possible zigzagging phenomenon in the solution procedure. The interested reader may refer to Sheffi (1985) for insights.

The second technique is referred to as the *boundary search procedure*, since it tends to choose solutions along the boundary, although it does consider interior points as well. The importance of the boundary search for origin-based assignment is that it is effective in eliminating residual flows, that is, small flows on sub-optimal routes. The elimination of residual flows is critical for algorithm convergence (see (Bar-Gera, 2002) for details). When the magnitude of step size is considered, it follows:

$$\alpha_{a\to b,\lambda}(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}') = \begin{cases} \left\{ \min\left(\alpha_a, \lambda \cdot \dfrac{x_{a\to b}(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}')}{q_j(\boldsymbol{\alpha})} \right) \right\} & q_j > 0 \\ \{\alpha_a\} & q_j = 0; \mu_a > \mu_b \quad \forall a, j \in N \setminus \{r\} \\ [0, \alpha_a] & q_j = 0; \mu_a = \mu_b \end{cases} \tag{23}$$

Note that when step size is set to 1, the above expression reduces to expression (20).

In order to guarantee descent of the objective function, and convergence of the algorithm, the search considers step size values of 1, 1/2, 1/4, 1/8, etc. The stopping condition is based on the concept of *social pressure*, $\Delta\mathbf{x}\cdot\mathbf{c}(\mathbf{x}+\lambda\cdot\Delta\mathbf{x})$, introduced by Kupsizewska and Van Vliet (1999). The basic idea is that every traveler shifted from route p_1 to p_2 applies pressure (positive or negative), which is equal to his /her gain (or loss) according to the difference in route costs that result from the shift. The total social pressure is the sum of the pressure from all the travelers. Our search direction is good in the sense that it always enjoys positive social pressure for small step sizes. As the step size increases, the social pressure decreases, and eventually it may become negative. Our goal is to find the largest step size—i.e., the first in the sequence 1, 1/2, 1/4, 1/8, etc.—with positive social pressure, i.e.,

$$\Delta\mathbf{x}\cdot\mathbf{c}(\mathbf{x}+\lambda\cdot\Delta\mathbf{x}) > 0 \tag{24}$$

This social pressure principle is in fact equivalent to the stopping condition of the line-search in the FW algorithm, except that this principle is applicable in certain cases in which the line-search optimization rule is not.

With the step size λ, aggregation of change of approach proportions is determined by the following formulas

$$\alpha_{j:b}(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}') = \left\{ \Delta\boldsymbol{\alpha} : \begin{array}{ll} \Delta\alpha_a \in -\alpha_{a\to b}(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}') & \forall a \in NB_j \\ \Delta\alpha_b = \sum_{a\in NB_j} \Delta\alpha_a & \\ \Delta\alpha_{a'} = 0 & \forall a_h^{'} \neq j \end{array} \right. \tag{25}$$

$$\alpha_j(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}' : A^r) = \bigcup_{\substack{b\in A^r : b_h = j;\\ \mu_b \le \mu_a, \forall a\in A^r ; a^h = j}} \alpha_{j:b,}(\boldsymbol{\alpha},\mathbf{c},\mathbf{c}') \tag{26}$$

3.3 Solution Algorithm

Based on the aforementioned discussion, the OBTAIN algorithm can be formally described as follows:

Step 0: Initialization.
For each origin r, let A^r be a tree of minimum-cost routes under free flow conditions from r. Let $\alpha_a^r = 1, \forall a \in A^r$ and $\alpha_a^r = 0, \forall a \notin A^r$. Set iteration counter n=0.

Step 1: For every origin r, *update restricting subnetwork* A^r.
Step 1.1: Remove unused links from A^r.
Step 1.2: Update data structures. For every node i, compute the maximum cost u_i from r to i. For every link $a=[i,j]$, if $u_i < u_j$, add link a to A^r.
Step 1.3: Find new topological order for new A^r. Update data structures.
Step 2: For every origin r, *update origin-base approach proportions* α_a^r.
Step 2.1: Compute average costs by (13).
Step 2.2: Compute Hessian approximations by (18).
Step 2.3: Compute flow shifts by (21).
Step 2.4: Determine the step size by (24).
Step 2.5: Project and aggregate flow shifts by (25) and (26).
Step 3: Convergence Check. If social pressure is positive, i.e., $\Delta \mathbf{x} \cdot \mathbf{c}(\mathbf{x} + \lambda \cdot \Delta \mathbf{x}) > 0$, then stop, otherwise apply flow shifts and update total link flows and link costs. Go to step 1.

4. NUMERICAL EXAMPLE

To demonstrate, we solve for a SCNE problem. The test network (shown in Figure 1) consists of two sectors. Each sector contains two agencies, i.e., manufacturers (indexed by i=1,2) and retailers (indexed by j=3,4). In addition, K represents the super-origin of the test network and dummy nodes 5, 6 are created for nodes 3 and 4, respectively.

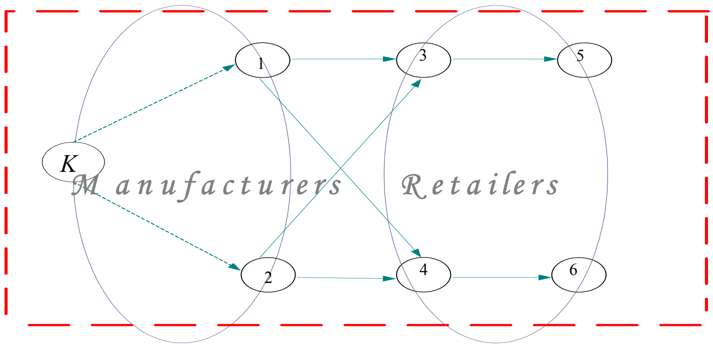

Figure 1. Supply Chain Network

The fixed demands for the two retailers are the same: $q_{K5} = q_{K6} = 33.216$. The cost functions associated with the two tiers of the test network are given in Table 1.

Table 1. Cost Functions for the SCNE problem with Two Tiers

Cost Function	Production /Handling and Storage Related Costs	Transaction Related Costs
M	$C_{K->1}(X^1) = 2.5x_{K->1}^2 + x_{K->1}x_{K->2} + 2x_{K->1}$ $C_{K->2}(X^1) = 2.5x_{K->2}^2 + x_{K->2}x_{K->1} + 2x_{K->2}$	$C_{ij}(x_{ij}) = 0.5x_{ij}^2 + 3.5x_{ij}$
	$c_{K->1}(X^1) = 5x_{K->1} + x_{K->2} + 2$ or $c_{K->2}(X^1) = 5x_{K->2} + x_{K->1} + 2$	$c_{ij}(x_{ij}) = x_{ij} + 3.5$
	$c'_{K->i}(X^1) = 5$	$c'_{ij}(x_{ij}) = 1$
R	$C_{3->5}(x_{3->5}) = 0.5(x_{3->5})^2$ or $C_{4->6}(x_{4->6}) = 0.5(x_{4->6})^2$	--
	$c_{3->5}(x_{3->5}) = x_{3->5}$ or $c_{4->6}(x_{4->6}) = x_{4->6}$	--
	$c'_{3->5}(x_{3->5}) = 1$ or $c'_{4->6}(x_{4->6}) = 1$	--

Remarks: "*M*" denotes manufacturers, and "*R*" retailers

Assuming all agencies are profit maximizers, the SCNE problem can be formulated as follows.

$$\min_{\mathbf{h}\in\Omega} z(\mathbf{x}(\mathbf{h})) = \sum_{a\in A^r} C_a \tag{27}$$

where feasible region Ω is already defined in (2) and (3). By applying the OBTAIN algorithm, the link results are summarized in Table 2.

Table 2. Link Results

a (I)	α_a (II)	x_a (III)	q_{a_h} (IV)	c_a (V)	μ_a (VI)	σ_{a_h} (VII)	c'_a (VIII)	υ_a (VIIII)	ρ_{a_h} (X)
$K\rightarrow 1$	1	33.216	33.216	201.296	201.296	201.296	5	5	5
$K\rightarrow 2$	1	33.216	33.216	201.296	201.296	201.296	5	5	5
$1\rightarrow 3$	0.5	16.608	33.216	20.108	221.404	221.404	1	6	6
$2\rightarrow 3$	0.5	16.608	33.216	20.108	221.404	221.404	1	6	6
$1\rightarrow 4$	0.5	16.608	33.216	20.108	221.404	221.404	1	6	6
$2\rightarrow 4$	0.5	16.608	33.216	20.108	221.404	221.404	1	6	6
$3\rightarrow 5$	1	33.216	33.216	33.216	254.62	254.62	1	7	7
$4\rightarrow 6$	1	33.216	33.216	33.216	254.62	254.62	1	7	7

I: Link; II: Approach proportion; III: Link Flow; IV: Flow to node a_h; V: Link cost; VI: Average link cost; VII: Average cost to node a_h; VIII: Link cost derivative; VIIII: Average link cost derivative; X: Average cost derivative to node a_h.

The above test results show that the first order optimality conditions (10) and the sufficient conditions for restricted user equilibrium (11) are fully complied at nodes 3 and 4. The route results, summarized in Table 3, can be easily derived by backtracking from the link results.

Table 3. Route Results

Route, p	Route Proportion, χ_p	Route Flow, h_p	Route Cost, c_p	Route Cost Derivative, $c_p^{'}$
$K \rightarrow 1 \rightarrow 3 \rightarrow 5$	0.5	8.304	254.62	7
$K \rightarrow 2 \rightarrow 3 \rightarrow 5$	0.5	8.304	254.62	7
$K \rightarrow 1 \rightarrow 4 \rightarrow 6$	0.5	8.304	254.62	7
$K \rightarrow 2 \rightarrow 4 \rightarrow 6$	0.5	8.304	254.62	7

It is also noted that Wardrop first principle is also complied, as for each O-D pair all used route costs are equal and minimum.

5. CONCLUSION AND SUGGESTIONS

This paper illustrates the experience of applying OBTAIN method to a SCNE problem which is essentially addressed as the fixed demand traffic assignment problem. The results satisfy the first order optimality conditions and the sufficient conditions for restricted user equilibrium. It is also worth mentioning that the route information can be easily derived from the link results in the case of acyclic network structure. The specialized network structure makes OBTAIN method efficient. It is also verified that the route results comply with Wardrop first principle which requires that if the (homogeneous) product flow is positive, then the corresponding product price is minimum, no matter where the product comes from. The OBTAIN method can be further exploited with larger networks (such as Chicago network) or extensive computation (such as K shortest route problems).

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SUMMARY OF NOTATION

a : link designation
a_h : head of link a
a_t : tail of link a
A^r : set of links in the restricted network originated from r
c_a : travel cost for link a
c_a' : derivative of travel cost function for link a
c_p^{rs} : travel cost for route p between O-D pair rs

h_p^{rs} : flow on route *p* between O-D pair *rs*
h_{pj}^r : flow on subroute *p* from origin *r* to node *j*
$\mathbf{h}$: vector of route flow
i,j : node designation
lcn : last common node
N : set of nodes
N_0 : set of origins
$o(i)$: topological order of node *i*, o(*r*)=1
p : route designation
P^{rs} : set of routes between O-D pair *rs*
q_j : traffic flow to node *j*
q^r : traffic production from origin *r*
$\bar{q}^{rs}$: traffic demand between O-D pair *rs*
$\mathbf{q}$: vector of q^{rs}
r : origin designation
s : destination designation
$S(r)$: set of destinations associated with origin *r*
x_a : link flow on link *a*
x_a^r : link flow from origin *r* on link *a*
$x_{a\to b}$: desirable amount of flow (in units of flows, say vph) that should be shifted from *a* to *b* in order to equalize costs
$\boldsymbol{\alpha}$: set of origin-based approach proportions
$\Delta\alpha_a$: change in origin-based approach proportion on link *a*
α_a : origin-based approach proportion on link *a*
$\alpha_{a\to b}$: desirable amount of origin-based approach proportion that should be shifted from *a* to *b* in order to equalize costs
α_{a_j} : origin-based approach proportion on link *a* whose head is node *j*
λ : step size
γ_p^{rs} : proportion of flow on route *p* between O-D pair *rs*
$\chi_{i\to j}$: proportion of flow to node *j* through node *i*
μ_a : average cost for approach *a*
σ_j : average cost to node *j*
υ_a : approximated derivative of μ_a with respect to x_a
ρ_j : approximated derivative of σ_j with respect to q_j
u_i : maximum cost to node *i*
ε_υ : small positive constant
Ω : feasible region associated with the traffic assignment problem