Journal of the Eastern Asia Society for Transportation Studies, Vol. 6, pp. 1382 - 1397, 2005

EQUITABLE NETWORK DESIGN

Richard CONNORS
Senior Research Fellow
Institute for Transport Studies
Leeds University
Leeds,
United Kingdom
Fax: +44-113-343-5334
E-mail: rconnors@its.leeds.ac.uk

Agachai SUMALEE
Senior Research Fellow
Institute for Transport Studies
Leeds University
Leeds,
United Kingdom
Fax: +44-113-343-5334
E-mail: asumalee@its.leeds.ac.uk

David WATLING
Professor of Transport Modelling
Institute for Transport Studies
Leeds University
Leeds,
United Kingdom
Fax: +44-113-343-5334
E-mail: dwatling@its.leeds.ac.uk

Abstract: The Network Design Problem (NDP) refers to the optimization problem faced by a planner whose aim is to improve a transport network, drawing on limited resources. Though the NDP may lead to, for example, a set of tolls that maximise *social welfare*, often no consideration is made of the distribution of resulting benefits and costs across the population of travellers. We consider a network under probit stochastic user equilibrium (SUE) with elastic demand, disaggregated into multiple user classes with different values of time and link-specific tolls. We propose the Theil measure for quantifying equity, which can be incorporated either into the objective function or the constraints within the NDP. A sensitivity analysis of the SUE flows provides the basis for computing the Jacobian of the social welfare function and of the Theil measure. This allows gradient-based optimisation algorithms to be used in solving the NDP. Numerical examples are reported.

Key Words: Network Design Problem, Equity, Sensitivity Analysis

1. INTRODUCTION

There is a growing consensus in the transportation community that some form of road pricing will be a central element of the future transport planning policies of today's congested urban cities. Its potential for transport demand management at the aggregate level is difficult to question. However, there continues to be concerns over the inequity of such a system: *ceteris paribus*, without recycling of the revenues generated to appropriately compensate the loss in consumer surplus, road pricing will place a greater restriction on freedom to travel for those (car-owners) on a lower income, and hence lead to a greater loss in consumer surplus for that group. These latter concerns arise in the midst of an increasing emphasis more generally on the social aspects of transport, for example by the commission of prominent studies on matters such as the impact of transport on social exclusion (SEU, 2003).

The question then naturally arises of how to address such concerns within the transport planning process. Often, this issue is interpreted as a tension between the 'qualitative' (the recognition that difficult-to-quantify concepts such as equity are important) and 'quantitative' (where the modelling tools conventionally used for transport planning require a mathematical abstraction of the concepts involved). Yet this seems unproductive and is, in any case, a tangential issue: the question of whether quantitative models are useful for transport planning is a separate one, as is the issue of whether the transport planning process itself is 'equitable' and consults sufficiently with the public. The issue we wish to address here is: 'How equitable is a given transport policy measure?', and by knowing that: 'How can we design transport policies with equity in mind?'. Implicit in the former question is some notion of scale for 'equity'. Such a scale could, for example, be based on some notion of proportionate voting of a community for a policy option, or some other measure of popularity of a qualitative judgement (Mueller, 1986, Hudson and Jones, 1995). On the other hand, it could be based on a measure of how evenly are the benefits and costs of a policy measure distributed among a population. The latter is the kind of approach adopted in the present paper, though this is only intended to serve as an example: other measures may be accommodated within the approach.

In particular we shall incorporate a study of equity into one of the canonical problems of transport network analysis, the Network Design Problem (NDP). In the NDP, the planner has a number of policy tools at his/her disposal (represented in the mathematical form of discrete and continuous design parameters), with which to optimise some measure of social good, while anticipating the response of travellers to the policy stimuli. That is to say, the planner aims to indirectly influence travel patterns towards some more beneficial state. Our contribution is to extend the continuous NDP to include constraints on the (in-) equitable distribution of impacts from any such policy measure. For mathematical convenience, we achieve this by proposing a smooth quantitative scale for equity, and clearly our approach is unapologetically quantitative, yet this allows us ultimately to trade off *qualitatively* the aggregate benefits of a policy with the equitable distribution of the benefits.

2. MEASURING EQUITY

Equity, like the related concepts of justice, fairness and 'right', is not a simple concept. Different people have different concepts of equity and the aspects of equity that seem important depend on the particular context (Langmyhr, 1997). There are two dimensions of equity: the vertical and horizontal. The vertical dimension is related to the inequality of the cost and benefit distributions amongst the different user classes, which can be categorised either by the socio-economic group or the need for a transport service, e.g. disability. The horizontal dimension of equity concerns the distribution of costs and benefits amongst a class considered to be of equals. In transport, the horizontal equity can be linked to the inequality in the cost and benefit of travel between users from the same class but making different travel movements.

Before defining a measure of equity, the unit of observation for the equity impact must be defined. In the social context, the choice of the unit can be an individual or a collective unit such as a household, women, the elderly, the disabled, a region, etc. Of course, the decision of the unit will determine the dimension and context of the inequality measurement. In this research, the model formulated (see next section) allows us to define the unit of observation by the user classes (distinguished by, say, income group) or by travel movement, and so we shall restriction attention to inequity in these two dimensions.

With the defined unit of observation, the next task is to define the observation item. In measuring the equity impact, we focus on the inequality of the distribution of the consumer surplus. There are several measures of inequality, reflecting different perceptions that inherently attach different sets of weights to transfers at various points in a distribution. This can result in contradictory ranking of a given pair of distributions (see Kolm, 1969). From the various approaches to measuring inequality, we consider the statistical measures; these include range, measure of variation, Gini measure and Theil's entropy measure. These measures can be applied for the evaluation of inequality of any vector or distribution of observations. In this paper we adopt the Theil measure (Theil, 1967), as defined in the next section.

As stated above, in order to address both dimensions of equity we need to distinguish between different movements on the network *and* between different classes of user; we therefore need a network model that includes both notions. The purpose of the following section will be to introduce the network model, as well as the formal measure of equity.

3. DEFINITIONS AND NOTATION

We consider a road network represented by a directed graph consisting of *N* nodes, with *A* the set of connecting links. The demand matrix has entries q_m^{rs} representing the travel demand by user-class *m* from origin *r* to destination *s*, where $r, s \in N$. The classes may represent any feature by which travellers are distinguished, ranging from travel purpose to socio-economic group and income level. In the model considered in this paper, only an explicit representation is made of the car drivers' network; an aggregate alternative (see below) is offered which could reflect a decision not to travel or to travel by an alternative mode; we note, however, that the approach is readily extendable to include an explicit representation of the public transport network too (Uchida *et al*, 2005). All users travelling by private car interact indistinguishably from one another when on the network, but the user classes differ in the link costs they experience and this engenders class-specific route choice behaviour for each OD movement.

The flow on link $a \in \mathcal{A}$ of user-class *m* is x_a^m with corresponding link cost (specific to class *m*)

$$t_a^m = \tau_a^m + \beta^m t_a(x_a) \ . \tag{1}$$

Here, $t_a(x_a)$ is the common link cost, a function of the total flow on that link: $x_a = \sum_m x_a^m$. The parameter β^m is the value of time for user-class *m* and τ_a^m is a constant cost (toll) specific to this link and this user-class.

The set of paths connecting node *r* to node *s* is denoted K_{rs}: we assume that the same paths are available to all user-classes. The (binary) link-path incidence matrix, Δ^{rs}, with elements $\delta_{a,k}^{rs}$, denotes the links comprising each path connecting node *r* to node *s*.

The path costs for class *m* are simply the summed constituent link costs:

$$c_k^m(f_k) = \sum_{a \in A} t_a^m \delta_{a,k} = \sum_{a \in A} \left[\tau_a^m + \beta^m t_a(x_a)\right] \cdot \delta_{a,k} \,, \tag{2}$$

with $f_k = \sum_a x_a \delta_{a,k}$ the total flow on path *k*. For user-class *m*, the perceived cost (travel time) on the *a*-th link is $T_a^m = t_a^m + \varepsilon_a^m$, a random variable that is centred on the deterministic link cost for this class, t_a^m, with the random errors ε_a^m normally distributed about 0 and (though unnecessary) independent across links. This defines the probit model, which naturally results in the network topology being reflected in the correlation structure of the perceived path costs. Each class has its own variances for the $\{\varepsilon_a^m\}$.

The proportion, $P_{k,m}^{rs}(\mathbf{c}^m) = \Pr(C_k^m < C_j^m \; \forall j \neq k \mid \mathbf{c}^m)$, of drivers from class m choosing the k-th path to travel from r to s are those who perceive this is the cheapest (given the current mean path costs for their class: $\mathbf{c}^m$). The Stochastic User Equilibrium (SUE) is defined to be a set of flows such that:

At SUE, no driver can improve their *perceived* travel cost by unilaterally changing route.

The SUE is the solution to the fixed-point problem:

$$\mathbf{f} = \mathbf{P} \cdot \mathbf{q}, \tag{3}$$

for path flows **f**, path choice probabilities **P**, and OD demands **q**. For each constituent OD movement, $\mathbf{f}^{rs} = P^{rs} \cdot \mathbf{q}^{rs}$, this comprises

$$\begin{bmatrix} f_0^{rs} \\ f_1^{rs} \\ \vdots \\ f_K^{rs} \end{bmatrix} = \begin{bmatrix} P_{0,1}^{rs} & P_{0,2}^{rs} & \cdots & P_{0,M}^{rs} \\ P_{1,1}^{rs} & P_{1,2}^{rs} & & \\ \vdots & & \ddots & \\ P_{K,1}^{rs} & P_{K,2}^{rs} & & P_{K,M}^{rs} \end{bmatrix} \cdot \begin{bmatrix} q_1^{rs} \\ q_2^{rs} \\ \vdots \\ q_M^{rs} \end{bmatrix}, \tag{4}$$

for the K paths connecting this OD pair and the M classes. This equilibrium condition is calculated in terms of the total link flows.

To determine the multinomial choice probabilities we follow Rosa (2003) and use the method of Mendell and Elston (1974). To calculate the probit-SUE link flows, we use the search direction from the MSA algorithm (toward the auxiliary flows) but calculate the step length using a combination of quadratic approximation (Rosa, 2003) and a line search to ensure descent at each iteration. The resulting aggregate link flows uniquely determine the corresponding class-specific link flows (as noted in Rosa, 2003).

3.1 Elastic Demand

For each OD pair, the option of "no travel" is represented in the network by a pseudo-link that provides drivers with another choice of OD path. The cost on this pseudo-link represents the opportunity cost of not travelling, or equivalently, the foregone utility that would have been gained by travelling to the destination. Following the probit model, the perceived cost of not travelling is $C_0 = c_0 + \varepsilon_0$, where on this link the random term can be considered to represent unobservable or un-measurable factors of utility. The utility of an OD movement for an individual is the benefit gained from travelling, or more typically, from arriving at the destination. It is only for analytical convenience that this utility appears as the cost on the pseudo-link in this modelling framework.

3.2 Social Welfare

Taking a utility-maximisation approach (rather than the equivalent cost minimisation approach), the net benefit gained by travelling is the utility minus the cost of travel: $C_{0,m}^{rs} - C_{k,m}^{rs}$ for users of class *m*, travelling on OD movement *r-s* by path *k*. The behavioural model underlying probit assignment assumes that users maximise utility, hence elect to travel on the path with minimum perceived cost, so that the expected indirect utility experienced by individuals (in user-class *m* travelling from *r* to *s*) is

$$U_m^{rs} = E\left[C_{0,m}^{rs} - \min_k \left\{C_{k,m}^{rs}\right\}\right] = c_{0,m}^{rs} - S_m^{rs}, \tag{5}$$

where c_0 is the (mean) pseudo-link cost and S_m^{rs} is the satisfaction (the expected minimum cost) for this OD movement for class *m*. This measure has an economic interpretation related to consumer surplus. For example, in the absence of income effects, a change in price or any other characteristics of the travel environment results in an expected change in consumer's surplus that is equal to the change in the above welfare measure. In our case, the consumer surplus is evaluated at the route choice level. The total consumer surplus is

$$CS = \sum_{rs}\sum_{m} q_m^{rs} U_m^{rs} = \sum_{rs}\sum_{m} q_m^{rs}\left(c_{0,m}^{rs} - S_m^{rs}\right). \tag{6}$$

The evaluation of social welfare (*SW*) involves two components, the consumer surplus and operator benefit:

$$SW = \text{consumer surplus} + \text{operator benefit}. \tag{7}$$

The operator benefit can be defined as the net financial benefit of the scheme: scheme revenue – scheme cost. Thus, the social welfare measure can be defined as:

$$SW = \sum_{rs}\sum_{m} q_{rs}^{m} U_{rs}^{m} + \sum_{a\in A_\tau} \tau_a x_a - \gamma\left|A_\tau\right| \tag{8}$$

with $A_\tau \subset A$ the index set of tolled links and γ the fixed cost of implementing a toll on one link.

3.3 Equity

As with modelling any social concept, there are several legitimate methods for quantifying equity of the network; in this paper we use Theil's entropy measure, which is commonly used in socioeconomics as a measure of inequality and in bioscience as a measure of diversity (it is equivalent to the Shannon index). Our methodology could equally be applied to other measures of inequality such as the Gini index. However, the Gini index is insensitive to changes in the extremes of the population distribution, and suffers from technical complications when trying to disaggregate contributions to the overall inequality from within and between constituent groups. The Theil measure does not suffer from these difficulties and can easily be extended to a more disaggregate analysis.

The Theil measure originates in information theory, where the entropy of a system can be defined as the average information content of that system. Theil's entropy measure is the difference between the entropy for the actual distribution (of income or any other values) and the entropy measure of the equal distribution (Theil, 1967). The basic formula of Theil's entropy is as follows:

$$T = \frac{1}{N}\sum_{i=1}^{N}\frac{Y_i}{\overline{Y}}\log\left(\frac{Y_i}{\overline{Y}}\right) \tag{9}$$

where N is the total population size, Y_i is the net economic benefit for group i and $\overline{Y}$ is the average net economic benefit across the whole population.

Aggregating the horizontal and vertical dimensions, with U_m^{rs} the (expected) utility of users from class m travelling from r to s, Theil's entropy measure for the network can be written as:

$$T = \frac{1}{\sum_{rs}\sum_{m} q_m^{rs}}\sum_{rs}\sum_{m}\frac{U_m^{rs}}{\overline{U}}q_m^{rs}\log\left(\frac{U_m^{rs}}{\overline{U}}\right) = \frac{1}{Q\overline{U}}\left(\sum_{rs}\sum_{m} q_m^{rs}U_m^{rs}\log\left[U_m^{rs}\right]\right) - \log\left[\overline{U}\right] \tag{10}$$

where $Q = \sum_{rs}\sum_{m} q_m^{rs}$ is the total demand across all OD movements and classes. For two classes, the Theil measure as a function of utility is show in Figure 1. Recall that $T = 0$ represents equality.

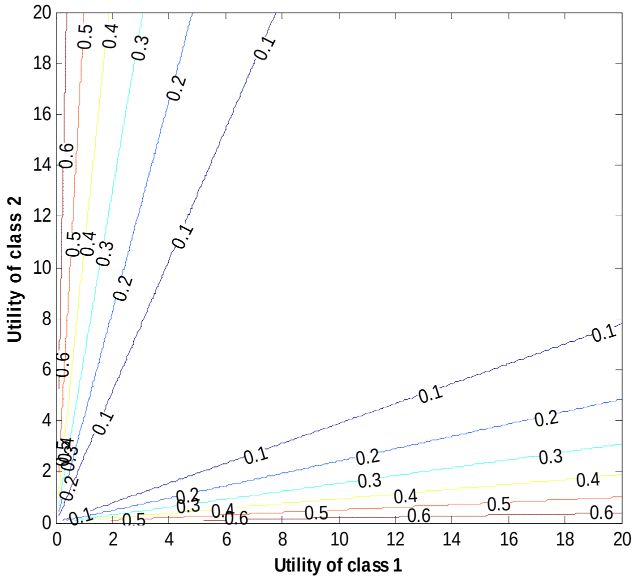


Figure 1: Contour Plot of the Theil Measure for Two Classes

4. THE NETWORK DESIGN PROBLEM

The NDP has a long history in the transport research literature, dating back at least to Abdulaal & Leblanc (1979). Using total system cost for capacity expansion as their objective function, a UE network model, and adopting the implicit function method, they demonstrated the inherent non-differentiability of the implicit problem, precluding the use of standard gradient-based numerical search methods. Since that time a variety of algorithms of increasing sophistication have been proposed to address this issue, for what is now commonly referred to as a bi-level program (Yang & Bell 2001, Patriksson & Rockafellar 2002) or MPEC, Mathematical Program with Equilibrium Constraints (Luo *et al*, 1996).

Our approach, however, has what turns out to be a significantly different point of departure from the work above. As described above, our network model is based on the probit SUE model. We believe that a strong case can be made for the superiority of this model to UE on behavioural grounds, in representing the many unobserved and heterogeneous elements of utility that motivate travellers' choice decisions. However, the probit SUE has a particular mathematical property in its favour in the context of the NDP, in that it smoothes out the non-differentiability of the implicit function that causes so many difficulties in the UE case (see Connors *et al*, 2004, for a more detailed discussion of this issue). Therefore, although the problem of non-convexity (and possibly multiple local optima) remains, we at least have the chance to employ standard gradient-based local search methods.

The version of the NDP we consider here addresses changing the class specific tolls (the vector of network design parameters **s**) in order to maximise the social welfare while the Theil measure of equity is maintained below some given value, λ, and the network flows are at equilibrium:

$$\max_{\mathbf{s}}\{SW[\mathbf{x},\mathbf{s}]\} \text{ such that } T(\mathbf{x})<\lambda \text{ and } \mathbf{x}=\tilde{\mathbf{x}}(\mathbf{s}), \tag{11}$$

where $\tilde{\mathbf{x}}(\mathbf{s})$ denotes the total link flows at SUE given design parameters **s**. In this paper we use the implicit function representation of the NDP

$$\max_{\mathbf{s}}\{SW[\tilde{\mathbf{x}}(\mathbf{s})]\} \text{ with } T(\tilde{\mathbf{x}}(\mathbf{s}))<\lambda\,. \tag{12}$$

4.1 Sensitivity Analysis

Standard optimisation algorithms need gradient information to find minima/maxima. For the NDP (12), computation of the relevant Jacobians by numerical differencing requires many SUE evaluations and is prone to error due to the amplification of inaccuracies in the calculation of the SUE flows (see Connors *et al.* 2004). A sensitivity analysis of the SUE flows, from (3), provides analytical expressions for the Jacobian of link flows and in turn, as we show in this section, gives Jacobian matrices describing the gradients of the objective function (see Magnanti & Wong, 1984) and the equity constraint.

Sensitivity analysis for the elastic demand UE case can be found in Yang (1997), for logit SUE in Davis (1994) and for single user class fixed demand probit SUE in Clark & Watling (2002). In this section we calculate the sensitivity expressions for the multiple user-class elastic demand probit SUE flows, and hence for the social welfare function, (8), and Theil measure (10).

The set of network design parameters, **s**, can include changes to the common link cost functions, to the class specific link constants, or to the class specific OD demands. In this paper we restrict our analysis to the case of tolling some links of the network, and this occurs in the model through the class-specific link constants, denoted τ_a^m in (1). From the SUE fixed point condition, (3), we define the distance function

$$\mathbf{d}(\mathbf{x},\mathbf{s})=\mathbf{x}-\Delta\cdot\mathbf{P}(\Delta^T\cdot\mathbf{t}[\mathbf{x},\mathbf{s}])\cdot\mathbf{q}\,. \tag{13}$$

This can be expressed as a sum of contributions over the OD pairs

$$\begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} - \sum_{rs}\Delta_{rs}\cdot\begin{bmatrix} P_{1,1}^{rs} & P_{1,2}^{rs} & \cdots & P_{1,M}^{rs} \\ P_{2,1}^{rs} & & & \\ \vdots & & \ddots & \\ P_{K,1}^{rs} & P_{K,2}^{rs} & & P_{K,M}^{rs} \end{bmatrix}\cdot\begin{bmatrix} q_1^{rs} \\ q_2^{rs} \\ \vdots \\ q_M^{rs} \end{bmatrix}. \tag{14}$$

The Taylor expansion, to first order, of this function about the SUE solution at $\mathbf{s}_0$ (having flows $\tilde{\mathbf{x}}(\mathbf{s}_0)$) gives

$$\mathbf{d}(\mathbf{x}(\mathbf{s}),\mathbf{s}) \approx \mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0),\mathbf{s}_0) + \left[\nabla_{\mathbf{x}}\mathbf{d}\big|_{\mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0),\mathbf{s}_0)}(\mathbf{x}-\tilde{\mathbf{x}}(\mathbf{s}_0))\right] + \left[\nabla_{\mathbf{s}}\mathbf{d}\big|_{\mathbf{d}(\tilde{\mathbf{x}}(\mathbf{s}_0),\mathbf{s}_0)}(\mathbf{s}-\mathbf{s}_0)\right] \tag{15}$$

$$\mathbf{d}(\mathbf{x}(\mathbf{s}),\mathbf{s}) \approx \mathbf{J}_{\mathbf{x}}(\mathbf{x}-\tilde{\mathbf{x}}(\mathbf{s}_0)) + \mathbf{J}_{\mathbf{s}}(\mathbf{s}-\mathbf{s}_0) \tag{16}$$

which defines the 'link flow Jacobian' $\mathbf{J}_{\mathbf{x}}$ and the 'design parameter Jacobian' $\mathbf{J}_{\mathbf{s}}$, that are evaluated at the equilibrium flows. Evaluating $\mathbf{d}(.,.)$ with the network flows at (the new) equilibrium, $\mathbf{x}=\tilde{\mathbf{x}}(\mathbf{s})$, gives $\mathbf{d}(\tilde{\mathbf{x}},\mathbf{s})=\mathbf{0}$ by definition of the distance function, so that the equilibrium flows at $\mathbf{s}$ can be expressed in terms of those at $\mathbf{s}_0$ as

$$\tilde{\mathbf{x}}(\mathbf{s}) \approx \tilde{\mathbf{x}}(\mathbf{s}_0) - \mathbf{J}_{\mathbf{x}}^{-1}\mathbf{J}_{\mathbf{s}}(\mathbf{s}-\mathbf{s}_0). \tag{17}$$

The link flow Jacobian is

$$\mathbf{J}_x = \mathbf{I}_n - \sum_{rs}\Delta_{rs}\cdot\sum_m q_{rs}^m \beta^m \nabla_{\mathbf{c}^m}\mathbf{P}_{rs}^m \cdot \Delta_{rs}^T \cdot (\nabla_{\mathbf{x}}\mathbf{t}) \tag{18}$$

where $\nabla_{\mathbf{c}^m}\mathbf{P}_{rs}^m$ is the path choice probability Jacobian (with respect to the path costs, $\mathbf{c}^m$) and $\nabla_{\mathbf{x}}\mathbf{t}$ is the Jacobian of common link costs with respect to the total link flows; with separable link cost functions this Jacobian is diagonal.

In the general case, the derivative term $\nabla_{\mathbf{s}}\mathbf{d}$ can be decomposed into contributions from those network design parameters in the vector $\mathbf{s}$ that correspond to changes in the common link cost function parameters, the class specific link constants, the values of time and the class specific OD demands. Here we only consider tolls that are imposed via the class specific link constants.

For design parameters changing the class specific link constant (i.e. a class specific toll),

$$\begin{bmatrix} \frac{\partial d_1}{\partial s_1} & \cdots & \frac{\partial d_1}{\partial s_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial d_N}{\partial s_1} & \cdots & \frac{\partial d_N}{\partial s_L} \end{bmatrix} = -\sum_i \Delta_i \sum_m \begin{bmatrix} \frac{\partial P_1^m}{\partial c_1^m} & \cdots & \frac{\partial P_1^m}{\partial c_k^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial P_k^m}{\partial c_1^m} & \cdots & \frac{\partial P_k^m}{\partial c_k^m} \end{bmatrix} \cdot \Delta_i^T \begin{bmatrix} \delta_{1,1}^m & \cdots & \delta_{1,L}^m \\ \vdots & & \vdots \\ \delta_{N,1}^m & \cdots & \delta_{N,L}^m \end{bmatrix} \cdot q_i^m \tag{19}$$

where $\delta_{n,l}^m = 1$ if the design parameter s_l corresponds to τ_n^m. Writing these indicator variables in the binary matrix $\nabla_{\mathbf{s}}\tau$:

$$\mathbf{J}_{\mathbf{s}} = -\sum_{rs}\sum_m q_{rs}^m \left(\Delta_{rs}\cdot\nabla_{\mathbf{c}^m}\mathbf{P}_{rs}^m \cdot \Delta_{rs}^T\right)\cdot\nabla_{\mathbf{s}}\tau \tag{20}$$

The Jacobians $\mathbf{J}_{\mathbf{x}}$ and $\mathbf{J}_{\mathbf{s}}$ give us the sensitivity of the SUE total link flows to the network design parameters and hence provide gradient information for the NDP optimisation. From the sensitivity analysis it easily follows that

$$\nabla_{\mathbf{s}}\tilde{\mathbf{x}}(\mathbf{s}) \approx -\mathbf{J}_{\mathbf{x}}^{-1}\mathbf{J}_{\mathbf{s}}\mathbf{I} \tag{21}$$

where $\mathbf{I}$ is the identity matrix (of size equal to the number of design parameters). Although this sensitivity analysis is conducted with respect to the total link flows, following Rosa (2003) it is clear that the disaggregated link flows are uniquely defined by the SUE assignment, and the sensitivity of the class-specific link flows to the design parameters follows from the sensitivities of the total link flows. We return to this later.

4.2 Gradient of the Social Welfare

To facilitate the optimisation required to solve the NDP, we require gradients of the objective function (social welfare) and constraint (equity).

The gradients of the social welfare function are

$$\frac{\partial SW}{\partial \tau_a^w} = \sum_{pq}\sum_n q_n^{pq} \sum_k \frac{\partial U_n^{pq}}{\partial \tilde{c}_k} \frac{\partial \tilde{c}_k}{\partial \tau_a^w} + \tilde{x}_a^w + \sum_n \sum_b \tau_b^n \frac{\partial \tilde{x}_b^n}{\partial \tau_a^w}, \tag{22}$$

where, at this stage, we include all paths k, and $\tilde{c}_k$ is the path cost with flows at equilibrium. If we assume that the satisfaction for a given OD movement and class depends only on the path costs for that OD movement and that class, we get:

$$\frac{\partial U_n^{pq}}{\partial \tau_a^w} = \frac{\partial\left[c_0^{pq,n} - S_n^{pq}\right]}{\partial \tau_a^w} = \sum_{rs}\sum_g \sum_{k\in K_{rs}} - \frac{\partial S_n^{pq}}{\partial \tilde{c}_k^{rs,g}} \frac{\partial \tilde{c}_k^{rs,g}}{\partial \tau_a^w} = -\sum_{k\in K_{pq}} P_k^{pq,n} \frac{\partial \tilde{c}_k^{pq,n}}{\partial \tau_a^w} \tag{23}$$

where $P_k^{pq,n}$ is the path choice probability for path k on this (pq-th) OD movement for this (n-th) class. So now we consider the dependence of the equilibrium path costs on the toll:

$$\frac{\partial \tilde{c}_k^{pq,n}}{\partial \tau_a^w} = \sum_{b\in A} \delta_{b,k}^{pq}\left(\delta_{b,a} + \beta^n \frac{\partial t_b(\tilde{\mathbf{x}})}{\partial \tau_a^w}\right) = \sum_{b\in A} \delta_{b,k}^{pq}\left(\delta_{b,a} + \beta^n \sum_{i\in A} t_b'(\tilde{x}_i) \frac{\partial \tilde{x}_i}{\partial \tau_a^w}\right). \tag{24}$$

In the final summation, over all links i, if we restrict our attention to separable link cost functions then the only relevant link flow is that on the b-th link of the outer sum, which allows us to simplify this expression to give

$$\frac{\partial \tilde{c}_k^{pq,n}}{\partial \tau_a^w} = \sum_{b\in A} \delta_{b,k}^{pq}\left(\delta_{b,a} + \beta^n t_b'(\tilde{x}_b) \frac{\partial \tilde{x}_b}{\partial \tau_a^w}\right). \tag{25}$$

Here the link flows represent total link flows, aggregated across all user-classes. The sensitivity analysis of the fixed point condition (presented above) gives the dependence of the equilibrium link flows on the tolls so that

$$\frac{\partial \tilde{c}_k^{pq,n}}{\partial \tau_a^w} = \sum_{b\in A} \delta_{b,k}^{pq}\left(\delta_{b,a} + \beta^m t_b'(\tilde{\mathbf{x}})\left[-\mathbf{J}_{\mathbf{x}}^{-1}\mathbf{J}_{\tau_a^w}\right]\right). \tag{26}$$

Combining these formulae gives the gradient of the social welfare function with respect to the design parameter tolling class w on link a:

$$\frac{\partial SW}{\partial \tau_a^w} = \sum_{pq}\sum_n q_n^{pq} \sum_{k\in K_{pq}} \left\{P_k^{pq,n} \sum_{b\in A} \delta_{b,k}^{pq}\left(\delta_{b,a} - \beta^m t_b'(\tilde{\mathbf{x}})\left[\mathbf{J}_{\mathbf{x}}^{-1}\mathbf{J}_{\tau_a^w}\right]\right)\right\} + \tilde{x}_a^w + \sum_n \sum_{b\in A_\tau} \tau_b^n \frac{\partial \tilde{x}_b^n}{\partial \tau_a^w}. \tag{27}$$

The final sum includes only those links being tolled, but requires us to calculate the disaggregated sensitivities of these link flows with respect to the tolls. To do this, consider the vector of link flows for class m alone, $\mathbf{x}^m$, as a function of the vector of choice probabilities for class m alone, the $\mathbf{P}_m^{rs}$, (over all OD pairs)

$$\mathbf{x}^m = \sum_{rs} q_m^{rs} \Delta^{rs} \mathbf{P}_m^{rs}[\mathbf{c}(\mathbf{x})] \tag{28}$$

so that

$$\frac{\partial \mathbf{x}^m}{\partial \tau_a^w} = \sum_{rs} q_m^{rs} \Delta^{rs} \nabla_{\mathbf{c}} \mathbf{P}_m^{rs}[\mathbf{c}(\mathbf{x})](\Delta^{rs})^T \left\{\beta^m \nabla_{\mathbf{x}} \mathbf{t}\left(\mathbf{J}_{\mathbf{x}}^{-1}\mathbf{J}_{\tau_a^w}\right) + \delta_a\right\} \tag{29}$$

where δ_a is an indicator vector with a 'one' in the a-th place only. The class link flow sensitivities to the toll(s) can therefore be calculated from the sensitivity analysis of the aggregate flows (the $\mathbf{J_x}$ and $\mathbf{J_\tau}$ matrices). This provides the sensitivities for the links being tolled that are required in equation (27) to give the gradients of the social welfare function.

4.3 Gradient of the Theil Measure

Calculating the gradients of the Theil measure proceeds in a similar way:

$$\frac{\partial T}{\partial \tau_a^w} = \sum_{pq} \sum_n \frac{\partial T}{\partial U_n^{pq}} \frac{\partial U_n^{pq}}{\partial \tau_a^w}. \tag{30}$$

Differentiating the Theil measure with respect to the utilities, and using

$$\frac{\partial \overline{U}}{\partial U_n^{pq}} = \frac{\partial}{\partial U_n^{pq}} \left(\frac{1}{Q} \sum_{rs} \sum_m q_m^{rs} U_m^{rs} \right) = \frac{q_n^{pq}}{Q}, \tag{31}$$

gives

$$\frac{\partial T}{\partial U_n^{pq}} = \frac{q_n^{pq}}{Q\overline{U}} \left(\log\left[U_n^{pq}\right] - \frac{1}{Q\overline{U}} \sum_{rs} \sum_m q_m^{rs} U_m^{rs} \log\left[U_m^{rs}\right] \right). \tag{32}$$

Using the expression for $\partial U_n^{pq} / \partial \tau_a^w$ from the social welfare gradient calculation, see equations (23) and (26), we have the derivative of the Theil measure with respect to the class-specific link tolls:

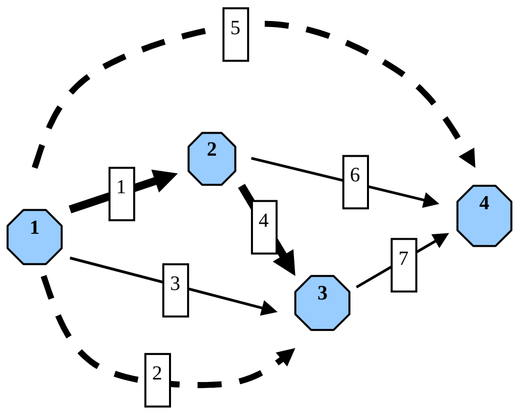
$$\frac{\partial T}{\partial \tau_a^w} = \frac{1}{Q^2 \overline{U}^2} \sum_{pq} \sum_n \left\{ \left(\sum_{rs} \sum_m q_m^{rs} U_m^{rs} \log\left[\frac{U_m^{rs}}{U_n^{pq}} \right] \right) \left(\sum_{k \in K_{pq}} P_k^{pq,n} \sum_b \delta_{b,k}^{pq} \left(\delta_{b,a} - \beta^m t_b'(\tilde{\mathbf{x}}) \mathbf{J}_\mathbf{x}^{-1} \mathbf{J}_{\tau_a^w} \right) \right) \right\}. \tag{33}$$

5. EXAMPLE

We consider the five-link network with elastic demand, two OD movements and two user classes with independent values of time. In Figure 2 below, some attempt has been made to represent the free flow travel time (shown by the length of the links) and susceptibility to congestion (shown by line width).

Links 2 and 5 are pseudo-links representing the no-travel option on the two OD movements. The (opportunity) cost of not travelling is fixed for these tests (as stated in the link cost function definitions in the figure); by definition of the common link cost functions, these values are multiplied by the value of time for each user-class. The link covariances are the same for each class: [0.3, 0.3, 0.15, 0.3, 0.3, 0.3, 0.15]. The class-specific link costs, (1), require the values of time β^m, and the tolls τ_a^m, to be specified for each test.

The network design problem we consider is to maximise the social welfare by tolling link 4 independently for each class, while keeping the Theil measure below some specified value (λ). The optimisation algorithm employed is the MATLAB function "fmincon" that is based on a sequential quadratic programming method that solves a quadratic programming (QP) subproblem at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration using the BFGS formula (Fletcher & Powell, 1963 and Goldfarb, 1970) and a line search is performed using a merit function similar to that proposed by Han (1977) and Powell (1978a, 1978b). The QP subproblem is solved using an active set strategy similar to that described in Gill *et al* (1981).



OD Movements: $1 \rightarrow 3$, $1 \rightarrow 4$

Common LCFs

$t_1(x) = 1 + x_1$ $\quad t_5(x) = 9$

$t_2(x) = 7$ $\quad t_6(x) = 5 + x_6^2$

$t_3(x) = 5 + x_3^2$ $\quad t_7(x) = 1 + x_7^2$

$t_4(x) = 1 + x_4 + \tau$

Figure 2: The Test Network

Several scenarios were examined, changing the values of time and the OD demands for each class and each OD movement. Illustrative results are shown below beginning with the base case where the classes are identical.

5.1 Equal Values of Time, Equal OD Demands for Each Class

Values of Time: $\beta^1 = 1, \beta^2 = 1$, demands: $q_1^{13}, q_2^{13} = 1$, $q_1^{14}, q_2^{14} = 1.5$. Plotted below against both social welfare and the Theil measure is a single optimisation 'run', maximising social welfare while maintaining T < 0.1. Since the values of time are equal, any inequality shown by the Theil measure arises from the horizontal dimension as the two OD movements are differentially affected by the tolls (note the class-symmetry in both plots).

Table 1. Optimal Tolls for 5.1

Theil Constraint	Initial Condition	Optimal tolls	Social Welfare
0.1	$\tau_4^1 = 3.0$ $\quad \tau_4^2 = 0.0$	$\tau_4^1 = 2.1342$ $\quad \tau_4^2 = 2.1342$	3.2938

Initial conditions $\tau_4^1 = 3.0, \tau_4^2 = 0.0$ (indicated by a circle) result in optimal tolls $\tau_4^1 = 2.1342 = \tau_4^2$ with a corresponding social welfare of 3.2938 (solid disc); the optimisation iterations are marked by asterisks, traced out, in order, by the dotted line. Despite the initial conditions violating the equity constraint ($\lambda = 0.1$), a valid optimal solution is found at which the constraint is not active. The contour plots in Figure 3 show that the Theil measure will not be an active constraint for (approximately) $\lambda > 0.08$. For smaller values of λ, there is a straightforward correspondence between increasing λ (relaxing the constraint) and attaining a higher value of social welfare at the optimal tolls.

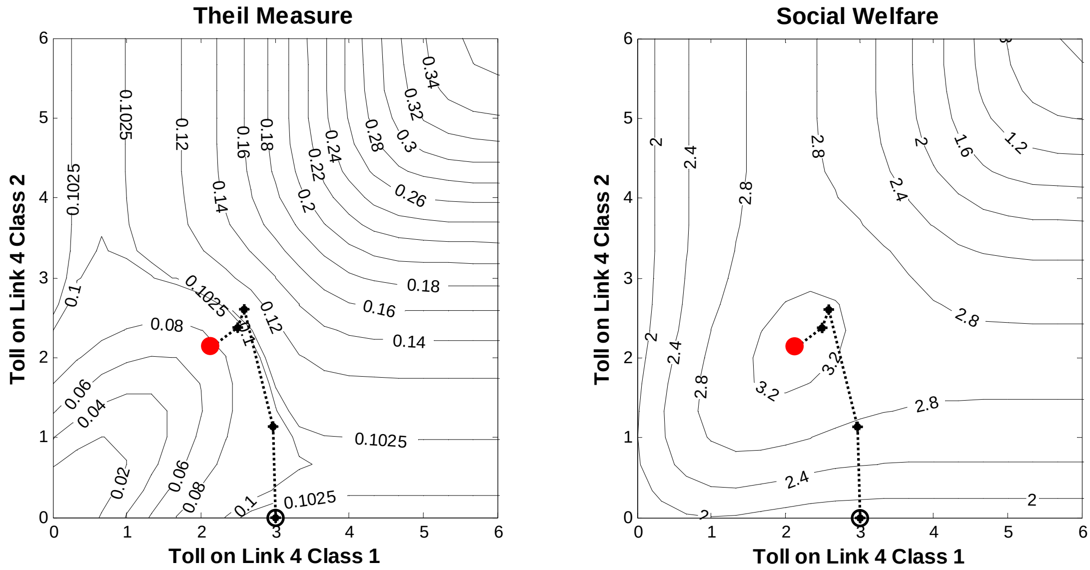


Figure 3: Theil Measure and Social Welfare Function for Example 5.1

5.2 Different Values of Time, Equal OD Demands for Each Class.

Different values of time: β^1=1, β^2=2, equal (by class) demands: q_1^{13}, q_2^{13} = 1.5, q_1^{14}, q_2^{14} = 1. Note the change (*cf.* 5.1) in the peak of social welfare, that extends into the region indicating a higher charge for class 1, whereas equity is maximised by a slightly higher charge for the higher value of time class 2. Optimisation for Theil < 0.12 shown with initial condition $\tau_4^1 = 0, \tau_4^2 = 1$ (indicated by a circle). The path of the optimisation is traced by a dotted line with each iteration marked by an asterisk, converging at $\tau_4^1 = 2.7688, \tau_4^2 = 3.3957$ (solid disc) giving social welfare of 5.0499.

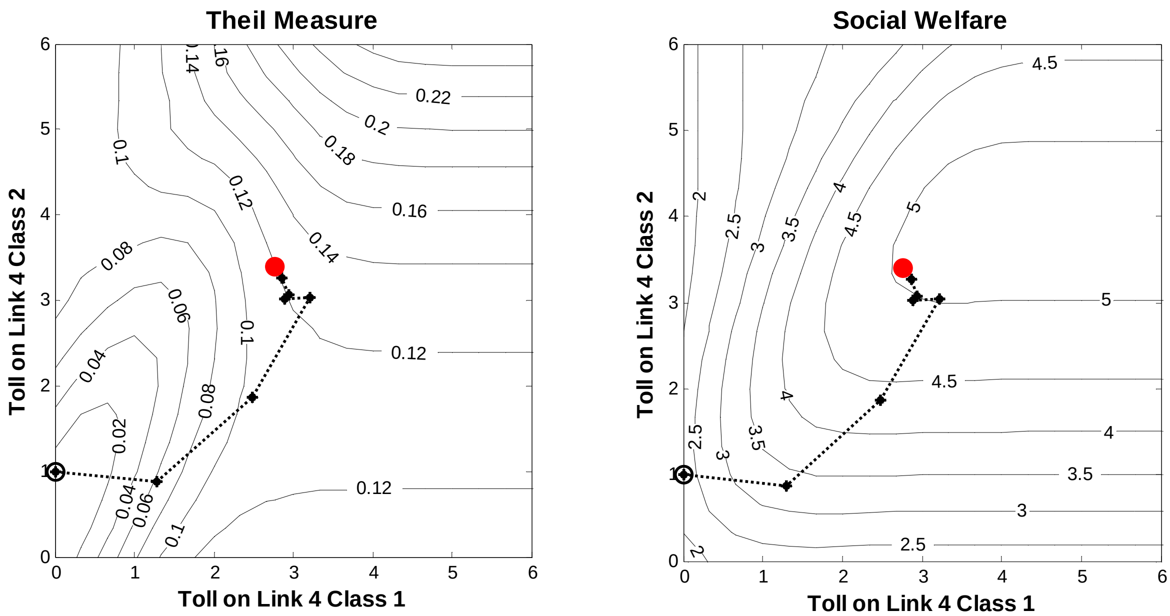


Figure 4: Theil Measure and Social Welfare Function for 5.2

The impact on social welfare of relaxing the equity constraint is shown in the table below:

Table 2. Optimal Tolls for 5.2

Theil Constraint	Initial Conditions	Optimal tolls	Social Welfare
0.06	τ_4^1=0.0 τ_4^2= 1.0	τ_4^1= 1.6905 τ_4^2= 2.6239	4.4430
0.12	τ_4^1=0.0 τ_4^2= 1.0	τ_4^1= 2.7688 τ_4^2= 3.3957	5.0499
0.18	τ_4^1=0.0 τ_4^2= 1.0	τ_4^1= 4.9563 τ_4^2= 3.9728	5.1712

5.3 Same Values of Time, Different OD Demands for Each Class.

Values of time: $\beta^1 = \beta^2 = 1$, as in Section 5.1, with lower OD demands for class 2 (as in Section 5.4): $q_1^{13} = 1, q_2^{13} = 0.5,\ q_1^{14} = 2,\ q_2^{14} = 1$. Note the change (*cf.* 5.1, 5.2) in the shape and location of the peak of social welfare, that now extends into the region indicating a higher charge for class 2, coinciding with maximum equity. It is clear that the equity constraint will not be active, even for values of λ as low as 0.02, and the global optimum of social welfare will be attained with near equal tolls on each class; optimisation for Theil < 0.1 is shown with initial condition $\tau_4^1 = 0, \tau_4^2 = 4$ (indicated by a circle), converging at $\tau_4^1 = 1.8744, \tau_4^2 = 1.8769$ (solid disc) giving social welfare of 2.6903.

Table 3: Optimal Tolls for 5.3

Theil Constraint	Initial Conditions	Optimal tolls	Social Welfare
0.1	τ_4^1=0.0 τ_4^2= 4.0	τ_4^1= 1.8744 τ_4^2= 1.8769	2.6903

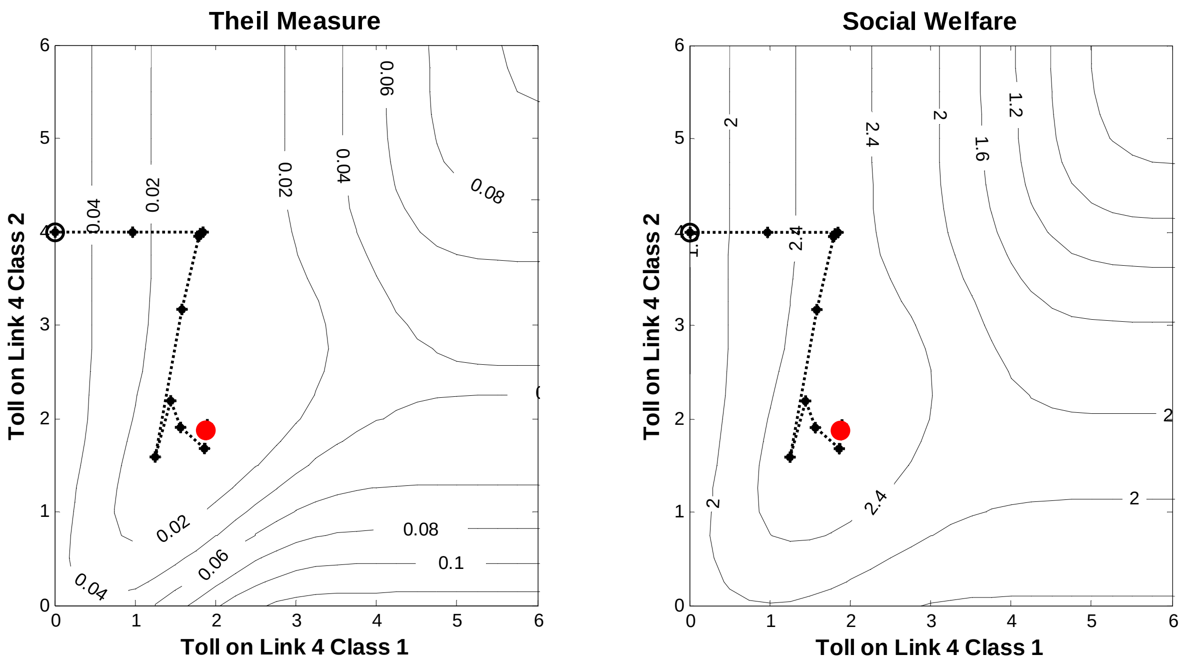


Figure 5: Theil Measure and Social Welfare Function for 5.3

5.4 Different Values of Time, Different OD Demands for Each Class

Values of time: $\beta^1=1, \beta^2=2$, demands: $q_1^{13}=1, q_2^{13}=0.5,\ q_1^{14}=2,\ q_2^{14}=1$. Class 2 now has less demand than class 1 on each of the two OD movements. The peak of social welfare has not moved appreciably, but the equity plot more strongly suggests (*cf.* 5.2) a higher charge for class 2.

Table 4. Optimal Tolls for 5.4

Theil Constraint	Initial Conditions	Optimal tolls	Social Welfare
0.05	$\tau_4^1=4.0 \quad \tau_4^2=7.0$	$\tau_4^1=3.0103 \quad \tau_4^2=3.5013$	3.8720
0.1	$\tau_4^1=0.0 \quad \tau_4^2=1.0$	$\tau_4^1=3.2444 \quad \tau_4^2=3.3773$	3.8848

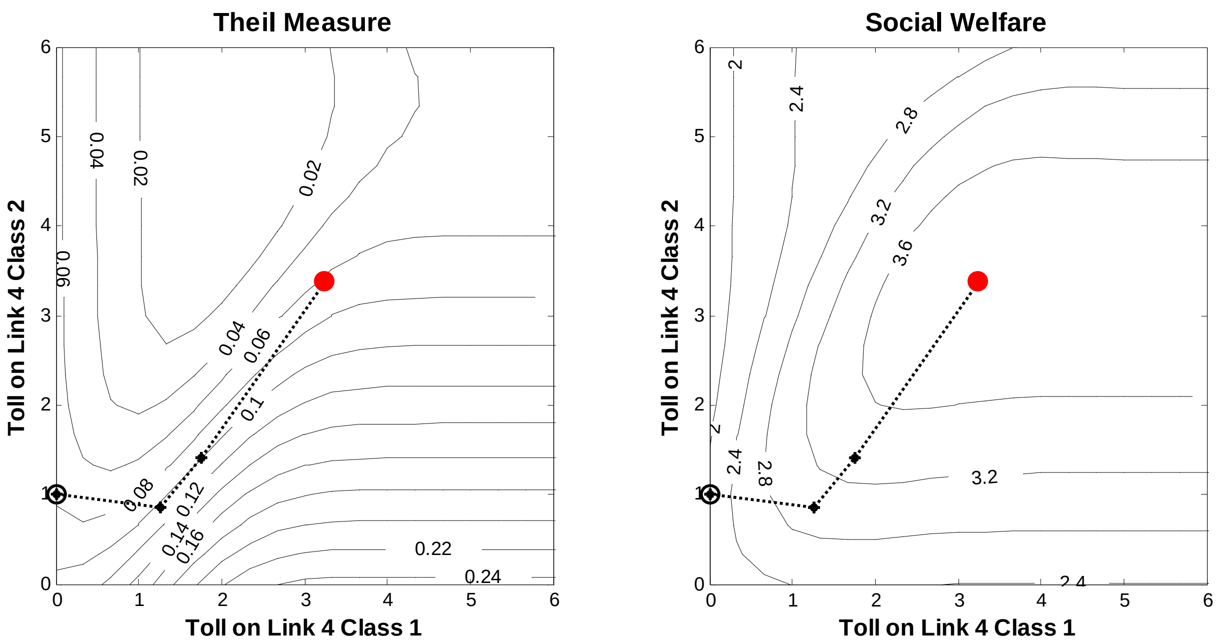


Figure 6: Theil Measure and Social Welfare Function for 5.4

6. CONCLUSION

In this paper we showed how the concept of equity can be quantified and hence included in the NDP as a constraint, and that the sensitivity analysis of the probit-SUE flows allows us to analytically derive gradients for the chosen equity measure. The resulting Jacobian matrices give information about the impact of local perturbations to the network design parameters and provide accurate and smooth gradient information to the optimisation algorithm. The numerical tests illustrate that this quantitative analysis allows a planner to make an informed qualitative trade off between equity and social welfare.

As discussed earlier, there are several legitimate methods for quantifying equity; our aim here is to show that this issue can be fully included in the analysis of the NDP, even when using probit-SUE with elastic demand and multiple user-classes for the network assignment. Theil's measure includes the inequality across the O-D movements and the user classes; these were aggregated in the analysis of this paper. Further work could investigate the consequences of separating the contribution from each dimension of equity by using the disaggregated Theil measure:

$$T = \sum_{rs} \left\{ T_{rs} \left(\frac{\bar{U}^{rs} q^{rs}}{Q\bar{U}} \right) \right\} + \frac{1}{Q} \sum_{rs} \left\{ q^{rs} \left(\frac{\bar{U}^{rs}}{\bar{U}} \right) \log \left(\frac{\bar{U}^{rs}}{\bar{U}} \right) \right\} \tag{34}$$

where, $\bar{U}^{rs}$ is the mean of consumer surplus across all classes for the rs-th O-D pair, and T_{rs} is the Theil's measure calculated within this group:

$$T_{rs} = \frac{1}{Q} \sum_{m} q_m^{rs} \left(\frac{U_m^{rs}}{\bar{U}^{rs}} \right) \log \left(\frac{U_m^{rs}}{\bar{U}^{rs}} \right). \tag{35}$$

The first term on the right hand side of (34) represents the within-group inequality, amongst different user classes from the same O-D movement. The second term represents the between-group inequality, the inequality between different O-D movements. Other disaggregate equity indices could be proposed based on (34): for example, contributions to the total inequity from 'short' and 'long' OD movements could be weighted in order to give greater importance to longer trips.

Further, we can consider related concepts such as the *acceptability* of a transport policy, using a similar approach to that presented in this paper, to analyse a class-dependent model of individuals' voting behaviour, in favour or against a given change to the network. A quantitative model of the acceptability of the proposed set of design parameters could be included into the NDP constraint alongside the equity measure presented in this paper.

ACKNOWLEDGEMENTS

The research described in this paper was conducted under the financial support of the U.K. Engineering & Physical Sciences Research Council. Thanks to Mike Maher and Andrea Rosa for valuable discussions regarding calculation of the probit SUE flows.

REFERENCES

Abdulaal, M. & Leblanc, L. (1979). Continuous equilibrium network design models. **Transportation Research 13B**, 19-32.

Clark, S.D. & Watling, D.P. (2002). Sensitivity analysis of the probit-based stochastic user equilibrium assignment model. **Transportation Research 36B**, 617-635.

Connors, R.D., Sumalee, A. & Watling, D.P. (2004). Understanding the Variable Demand Probit-based Network Design Problem. **36th Annual Conference, UTSG 5th-7th January 2004**.

Davis, G.A. (1994). Exact local solution of the continuous network design problem via stochastic user equilibrium. **Transportation Research 28B**, 61-75.

Fletcher, R. & Powell, M.J.D. (1963). A Rapidly Convergent Descent Method for Minimization, **Computer Journal 6**, 163-168.

Gill, P.E., Murray W. & Wright M.H. (1981). **Practical Optimization**, London, Academic Press.

Goldfarb, D. (1970). A Family of Variable Metric Updates Derived by Variational Means. **Mathematics of Computing 24**, 23-26.

Han, S.P. (1977). A Globally Convergent Method for Nonlinear Programming, **Journal of Optimization Theory and Applications 22**, 297.

Hudson, J. & Jones, P. (1995). The importance of the 'ethical voter': as estimate of 'altruism'. *In* Zamagni S. (eds.) **The Economics of Altruism**, Edward Elgar Publishing Limited, UK.

Kolm, S.C. (1969). The optimal production of social justice. *In* Margolis, J. and H. Guitton (eds.) **Public Economics**, McMillan, London.

Langmyhr, T. (1997). Managing equity: The case of road pricing. **Transport Policy 4(1)**, 25-39.

Luo, Z-Q., Pang, J-S. & Ralph, D. (1996). **Mathematical Programs with Equilibrium Constraints**. Cambridge University Press.

Magnanti, T.L. & Wong, R.T. (1984). Network Design and Transportation Planning: Models and Algorithms. **Transportation Science 18**, 1-55.

Mendell, N.R. & Elston, R.C. (1974). Multifactorial qualitative traits: genetic analysis and prediction of recurrence risks. **Biometrics 30**, 41-57.

Mueller, D.C. (1986). Rational egoism versus adaptive egoism as fundamental postulate for a descriptive theory of human behaviour. **Public Choice 51**, 3-23.

Patriksson, M. & Rockafellar, R.T. (2002). A Mathematical Model and Descent Algorithm for Bilevel Traffic Management. **Transportation Science 36(3)**, 271-291.

Powell, M.J.D. (1978a). A Fast Algorithm for Nonlinearly Constrained Optimization Calculations, **Numerical Analysis**, ed. G.A. Watson, Lecture Notes in Mathematics, Springer Verlag, Vol. **630**.

Powell, M.J.D. (1978b). The Convergence of Variable Metric Methods For Nonlinearly Constrained Optimization Calculations. *In:* O.L. Mangasarian, R.R. Meyer, and S.M. Robinson, eds. **Nonlinear Programming 3**, Academic Press.

Rosa, A. (2003). **Probit based methods in traffic assignment and discrete choice modelling.** Ph.D. Thesis, Napier University.

S.E.U. (2003). Making the Connections: Final Report on Transport and Social Exclusion. Social Exclusion Unit, Office of the Deputy Prime Minister, U.K.

Theil, H. (1967). **Economics and Information Theory**. North Holland Publishing Company, Amsterdam.

Uchida, K., Sumalee, A., Watling, D. & Connors, R. (2005). A Study on Optimal Frequency Design Problem for Multi-modal Network using Probit-based User Equilibrium Assignment. Paper to be presented at **Transportation Research Board 84th annual meeting,** January 9th–13th 2005, Washington DC..

Yang, H. & Bell, M.G.H. (2001). Transport bilevel programming problems: recent methodological advances. **Transportation Research 35B(1)**, 1–4.

Yang, H. (1997). Sensitivity analysis for the elastic-demand network equilibrium problem with applications. **Transportation Research 31B(1)**, 55-70.