Abstract: This paper proposes the methodology to evaluate the road network anti-hazardous ability by the concept of network capacity reliability. The methodology proposed here considers the area-traffic regulation, which may be implemented when a large disaster occurs in Japan. The model to calculate entry permission rates is formulated as MPEC (Mathematical Programming with Equilibrium Constraints) in which the travellers are assumed to rationally respond to the imposed permission rate according to the Wardrop’s equilibrium. Also in this paper, an efficient algorithm to find a solution of MPEC problem is proposed. The proposed method transforms the original bi-level optimisation problem into a smooth single level optimisation problem by utilising the idea of merit function. The efficiencies of proposed network reliability index together with calculation methodology are discussed by the numerical experiment.

Key Words: Network Capacity, Traffic Management, MPEC, Network Reliability

1. INTRODUCTION

Due to the Great Hansin-Awaji Earthquake, the road network suffered the destruction of expressways and bridges as well as blocking of major arteries, resulting in the closure of roads and congestion. This incident caused a severe problem for not only day-to-day travellers but also emergency vehicles under operation. One of the most crucial elements of traffic management or traffic regulation after a major disaster, such as an earthquake, is to maintain network performance during the period from the outbreak to its settlement. The experience from the Great Hansin-Awaji Earthquake suggested the necessity to look at the approach to maintaining the reliability of the network after the incident. For improving traffic condition after the disaster, authors have proposed two-stage area regulation method.

The problem of defining the optimal traffic regulation plan can be formulated as a Mathematical Programming with Equilibrium Constraints (MPEC). MPEC involves two classes of players in traditional Stackelberg game theory, the leader and the follower. In our case, the leader is the network manager who tries to set the traffic regulation plan (e.g. access control in different directions and/or zones) so as to maximise the reliability indices of the traffic network. At the same time, the regulation plan will affect the cost or ability (feasibility in some extreme cases) of travel on some routes. Thus, the follower (road user) will rationally change his or her route, according to the Wardrop’s user equilibrium principle (Wardrop,
1952) so as to minimise his or her travel cost, to a new user equilibrium point. It should be noted that in MPEC the leader is assumed to have the ability to anticipate the follower’s response to his or her policy.

It is well known that MPEC is one of the most challenging problems in optimisation. The equilibrium condition will be initially cast in its KKT condition in which a set of conditions is in the form of complementarity constraints (CP) which imposes unfriendly disjunctive characteristics to the optimisation problem. Thus, we will utilise the idea of a “merit function” to transform CPs to a semi-smooth non-linear constraint. A merit function is simply an equivalent non-linear equation of the CP. This formulation will allow us to exploit various existing optimisation methods which can deal with the semi-smooth constraint.

The structure of this paper is summarised as follows. First, the importance of traffic regulation in an emergent condition, and two-stage area regulation method are explained. Then, a model for selecting the entry permission rates are formulated as a bi-level optimisation problem. We review transport network reliability measures, which is useful to evaluate the anti-hazardous ability of the network. We will then look into the detail of the appropriate indices for the analysis of the regulation of traffic after a major disaster. Third, reformulation of the problem with smooth single level optimisation is explained. A solution algorithm based on the Sequential Quadratic Programming Algorithm is developed and used to find the optimal traffic regulation patterns. Finally, the efficiency of proposed algorithm is evaluated through the numerical experiment.

2. NETWORK CAPACITY AND TRAFFIC REGULATIONS

2.1 Lessons from Great Hanshin-Awaji Earthquake

The road network capacity deteriorates enormously when a large disaster occurs, as observed when the Great Hanshin-Awaji Earthquake occurred in 1995 in Japan. The road network suffered from the destruction of expressways and bridges, as well as the blockade of major arteries, resulting in the closure of roads and traffic jams. This caused serious difficulty in travelling, not only for ordinary cars, but also for emergency vehicles. There are many lessons from Great Hanshin-Awaji Earthquake which should be reflected onto the transport planning for the future. First, importance of the road network system increases enormously in an emergent condition. Comparing with rail systems, the road system is more robust against disasters because it has many alternatives of routes. Moreover, analysis of changes in the OD traffic volume for automobiles within road networks of the Hanshin region after the earthquake showed that the OD ratio of short-distance trips increased greatly (Kurauchi and Iida, 1998). This increase was directly related to activities such as purchasing goods and materials, and was generated in large part by the need to rely on automobiles due to a lack of availability of other modes of transportation normally used for everyday activities (IATSS, 2000). Not only does this make the absolute prohibition of passenger cars highly problematic, but in consideration of the fact that most of these trips were for very short distances, it argues for a policy that permits their use as much as possible. Second, a ‘linear’ regulation method such as settling emergency roads is not a sufficient method when a large disaster happens. After the Great Hanshin-Awaji Earthquake, the emergency routes that were exclusively used by rescue teams seemed to be free of congestion. Nevertheless, the access/egress roads to/from the emergency routes were heavily congested with passenger cars so the emergency vehicles needed a long time to reach their destinations. Third, we should evaluate the road network not only considering the normal condition but also considering the degraded condition. In designing infrastructures, we usually discuss how much anti-hazardous ability the infrastructure must have in order not to be destroyed. It can be said that these discussion concentrates on how to prevent the damages of infrastructures. Besides this, we should evaluate how much the consequent of damage influences onto the total functionality of the road network.
2.2 Traffic Regulation in an Emergent Condition

Corresponding to the lesson that a liner regulation method is not sufficient when a large disaster happens, the act efficient at emergency condition after a disaster, the Disaster Prevention Basic Act, is revised so as to regulate traffic within an entire affected area in addition to a linear regulation of traffic of the specific routes or roads. Considering the revision, the need is apparent for the development of contingency plans delineating the scope and method of implementation.

There can be a variety of methods to implement area traffic regulation. In this study, we adopt two-stage area regulation method proposed by Iida et al. (2000). The proposed scheme for the two-stage area regulation for Hanshin region is illustrated in Figure 1. The first-stage in the proposed regulation involves a strict entry prohibition for vehicular traffic from regions outside the damaged area except for emergency vehicles. The second-stage regulation executes specification of the cities stated above as regulation sub-areas, and prohibits the entry of traffic into these sub-areas. Under the second-stage area regulation, traffic could move freely both within and out of the sub-areas to unregulated areas. The former movement corresponds to the purchasing goods, and the latter corresponds to evacuation. Area regulation for limiting the number of vehicles allowed to enter a regulated zone can be effected by allowing only vehicles with valid permits to enter. Concrete methods for the implementation of area regulation require deciding control variables such as planning of dividing the area to be regulated into zones, and determining which vehicles are allowed to enter, depending on damage condition of road network. To decide these control variables, the concept of network capacity can be applied. In other words, we can decide the control variables by formulating a mathematical model to maximise total network capacity under the condition that there are not substantial delays on the network.

2.3 Mathematical Formulation for Selecting the Two-Stage Area Regulation

To embody the concept of two-stage area regulation method and help to decide the regulation policy, Iida et al. (2000) proposed a mathematical model to calculate entry permission ratio for first- and second-stage regulations. The proposed two-stage area regulation algorithm calculates both the first- and the second- stage regulation ratios to maximise generation and concentration of traffic volume within the regulated area, subject to acceptable level of congestion selected by a planner. The success of implementing the two-stage area regulation varies depending on factors such as the driver’s route choice behaviour or the extent to which dynamic route guidance information is provided to the drivers. For simplicity, let us assume that drivers select the shortest route in terms of travel time. Assuming that the traffic manager
has a complete control over the traffic with the 1st and the 2nd stage area regulation, the 2-stage area traffic regulation model can, thus, be formulated as shown below.

Upper Problem

\[
\max \left( \sum_{i \in C} \sum_{j \in C} \bar{\phi} \cdot \xi_{ij} \cdot OD_{ij} + \sum_{i \in C} \sum_{j \in C} \eta_{ij} \cdot OD_{ij} + \sum_{i \in C} \sum_{j \in C} (1 - \xi_{ij} - \eta_{ij}) \cdot OD_{ij} \right) \rightarrow \text{max (1)}
\]

s.t.

\[
0 \leq \bar{\phi}, \Phi \leq 1
\]

\[
\sum_{j \in C} x_a^j \leq kC_a \text{ for all } a \in L
\]

Lower Problem

\[
\sum_a \int_a^{\Sigma_{C}} t_a(x) \, dx \rightarrow \text{min (4)}
\]

s.t.

\[
\theta \xi_{ij} \cdot OD_{ij} + \phi \eta_{ij} \cdot OD_{ij} + (1 - \xi_{ij} - \eta_{ij}) \cdot OD_{ij} + \sum_{a \in L} \delta_{ia} x_a^j = 0
\]

\[
\text{for all } j \in C, i \in \{N - j\} (5)
\]

\[
x_a^j \geq 0 \text{ for all } a \in L, j \in C
\]

where

\[
\begin{align*}
\theta &: \text{entry permission rate for 1st regulation,} \\
\phi &: \text{entry permission rate for 2nd regulation,} \\
\xi_{ij} &: \text{if OD pair } ij \text{ is involved with 1st regulation, then 1, and otherwise 0,} \\
\eta_{ij} &: \text{if OD pair } ij \text{ is involved with 2nd regulation, then 1, and otherwise 0,} \\
OD_{ij} &: \text{Traffic volume between OD pair } ij \\
C &: \text{a set of concentration centroids,} \\
N &: \text{a set of nodes,} \\
L &: \text{a set of links,} \\
\delta_{ia} &: \text{is 1 if link } a \text{ is leading into node } i \text{ and is -1 if link } a \text{ is leading out of node } i. \text{ If link } a \text{ is not connected with node } i, \text{ then 0,} \\
k &: \text{allowable congestion level,} \\
C_a &: \text{capacity flow on link } a, \\
t_a(x) &: \text{travel time function on link } a, \\
x_a^j &: \text{flow on link } a \text{ with destination } j.
\end{align*}
\]

The formulation can be categories as a network design problem. The system designer, traffic manager is able to anticipate the rational responses from the road users in terms of their route choices. This situation is known as a Stackelberg game (Fisk, 1986). Even with the linear programs in both upper and lower level problems, the problem is proved to be an NP-Hard in which there does not exist any optimisation program being capable of obtaining the global solution. However, there are some algorithms which can guarantee the local optimum of this problem.

The solution algorithm applied in Iida et. al (2000) is a heuristic method called Complex (Constrained Simplex) Method (Box, 1965), which takes time to converge because every time the algorithm try a new vertex in the upper problem, the lower problem, user equilibrium assignment, should be solved.

2.4 Transport Network Reliability

To evaluate the anti-hazardous ability of transport network, we should consider the possible damage and its consequence. In this sense, the concept of transport network reliability is
adequate to evaluate the anti-hazardous ability of transport network. The transport reliability is a conceptual framework to create reliable and flexible transport systems, and a plenty of reliability measures have been proposed in response to researchers’ interests. Network reliability analysis can be roughly classified into two types; whether it considers unusual conditions such as when disaster happens, or it discusses rather usual conditions such as when traffic incidents or congestions happen.

Terminal reliability is commonly defined as “the probability that nodes are connected, i.e. it is possible to reach the destination” (Nicholson et al, 2003). Terminal reliability does not consider the delay by the congestion because it does not look at actual performance but at the intrinsic characteristics of the topology of the network. It is usually applied to evaluate for unusual condition. One reason is that delay is not important in such an emergent condition, and the fact nodes are connected, is a prior concern. However, the lesson from Hanshin-Awaji Earthquake suggests that connectivity reliability is not a sufficient measure even when an emergent condition.

Another well known measure of network reliability is called travel time reliability or performance reliability. This is commonly defined as “the probability that a trip can be successfully finished within a specified time interval” (Wakabayashi and Iida, 1992). This measure considers the quality of services, which terminal reliability does not consider. In general, this measure is used to evaluate the network for day-to-day fluctuation, in other words, usual condition.

In response to the fact that terminal reliability measure only considers the connectivity, capacity reliability has been proposed by Chen et al. (1999). This has been defined as “the probability that the network can accommodate a specific demand level”. This measure utilises the concept of network capacity, and calculates maximum OD matrix multiplier \( \mu \). By Monte Carlo simulation methods, they calculate the probability that multiplier \( \mu \) is larger than specific multiplier, \( \mu_r \).

One of the difficulties in evaluating the transport reliability is that we do not know precisely which links are degraded. This implies that there are large uncertainties in inputs. Therefore, most of the studies utilise the probabilistic approach. As the model gets more complex, it gets harder to evaluate the measures by mathematical transformation, and many reliability measures utilises Monte Carlo simulation methods. Although rapid advancement of computer technology makes it easier to apply this kind of iterative approaches, the calculation of transport reliability is, in general, a time consuming work.

Another unique approach is to assume the worst case scenario and evaluate the network (Bell, 2000). They assume that ‘evil entity’ tries to maximise the consequence of link blockade. Then, the network is evaluated by worst-case scenario. This methodology assumes that only one link is degraded because it is a ‘rare’ event. However, talking about disasters, many links are degraded spontaneously although this event happens very rarely.

2.5 Network Reliability Analysis Considering Traffic Regulation

So far, we discussed the importance of traffic regulation when a disaster happens, and the regulation influences largely on the flow of the traffic after the disaster. This means that we should consider the possible regulation strategy after the disaster in evaluating the anti-hazardous ability of transport network. The mathematical model explained in the previous section is a special case of the network capacity model. This implies that using this model, the capacity reliability measures can be calculated. By applying the concept of capacity reliability, we can evaluate the network capacity while considering the possible traffic regulation after the disaster. A possible set of scenarios for network disruption is created based on the likelihood of disruption, and the network is evaluated based on these scenarios.
As the proposed model is a computationally intensive method, recently, authors proposed the modified model assuming that traffic managers can control the routes to use for all drivers (Kurauchi, 2003). This strict assumption simplifies the model into a simple linear programming problem, although it would be unrealistic. On the contrary this study explores an efficient method to calculate original bi-level programming problem. We replace the demand problem by optimality conditions, and the conditions are transformed into smooth functions, which would be explained in detail at next section.

3. MATHEMATICAL REFORMULATION OF THE MODEL

3.1 Reformulation of Network Capacity Model to a Single Level Optimisation

The formulation of the network capacity model proposed in the previous section can be simplified as follows:

\[ \max_{y} f(x, y) \]

\[ \text{s.t. } y \in Y_{ad} \]

\[ x \in S(y) \]

where \( f : \mathbb{R}^{m+n} \to \mathbb{R} \) is a continuous differentiable function which is the objective of the traffic network regulator, \( Y_{ad} \subseteq \mathbb{R}^n \) is the feasible space for the entry permission rates, \( y \in Y_{ad} \) is a vector of the entry permission rates which can be decomposed as \( y = \begin{bmatrix} \theta \\ \phi \end{bmatrix} \). For each \( y \in Y_{ad} \) and for a continuous differentiable function \( t : \mathbb{R}^{m+n} \to \mathbb{R}^m \), which is a mapping of link travel time, \( S(y) \) is the solution set of the Wardrop’s user equilibrium condition (UE), mentioned earlier. The solution set of UE can be defined by the pair \((t(y, \cdot), \Psi(y))\) in which \( x \in S(y) \) if and only if \( x \in \Psi(y) \) and satisfies the following variational inequality:

\[ (v-x)^{T} \cdot t(x, y) \geq 0 \text{ for } \forall v \in \Psi(y) \].

In order to ensure the uniqueness of the solution to the VI, the assumption of the monotonicity of the travel time function following Smith (1979) is assumed. Strictly, it is assumed that \( t(\cdot) \) is a positive value, continuous, and monotonically increasing over \( \Psi(y) \). The latter condition is defined by the property:

\[ (t(v) - t(w))^{T} (v - w) > 0 \text{ for } \forall v, w \in \Psi(x) \text{ } (v \neq w) \]

In this paper, the feasible space of the vector \( x \) is defined on link-node space, see §2.2 (Patriksson, 1994) associated with other constraints. We assume, at this stage, that \( \Psi(y) \) is defined by:

\[ \Psi(y) := \{ x \in \mathbb{R}^n : g_i(x, y) \geq 0, i = 1, \ldots, l \} \]

with \( g : \mathbb{R}^{m+n} \to \mathbb{R}^l \) continuously differentiable and concave in the first variables. The blanket
assumption to be made is the constraint qualification of the linear independence of the partial
gradient $\nabla_i g_\cdot (x, y), i \in I(x, y)$ where

$$I(x, y) := \{ i : g_i(x, y) = 0 \}$$  \hspace{1cm} (11)

The next step is to reformulate the VI to the equivalent KKT condition. By assumption on the
monotonicity of the travel time function, there exists one and only one solution of the VI.
Furthermore, by the assumption of the linear independence of function defining $\Psi(y)$, every
solution for VI must satisfy the KKT conditions

$$t(x, y) - \nabla_i g(x, y) \lambda = 0, \ \ g(x, y) \geq 0, \ \ \lambda \geq 0, \ \ \lambda^T \cdot g(x, y) = 0.$$  \hspace{1cm} (12)

where $\lambda \in \mathbb{R}^l$ is a Lagrange multiplier which is uniquely determined. Next we prove the
equivalent of the KKT formulation and the VI. Let $F$ denote the feasible region of $MPEC-VI$
and for any $(x, y) \in F$, define $M(x, y)$ to be the set of all multiplier vectors $\lambda \in \mathbb{R}^l$ such that:

$$M(x, y) = \left\{ \lambda \in \mathbb{R}^l : t(x, y) - \nabla_i g(x, y) \lambda = 0, \ g(x, y) \geq 0, \ \lambda^T \cdot g(x, y) = 0 \right\}.$$  \hspace{1cm} (13)

Based on Luo et al (1996), Theorem 1.3.5, the $MPEC-KKT$ is equivalent to the $MPEC-VI$ if
and only if the sequentially bounded constraint qualification (SBCQ) for the $MPEC-VI$ is
hold. “The SBCQ is equivalent to saying that KKT conditions hold with bounded multipliers
on bounded sets” (Luo et al, 1996). The following lemma proves that our traffic regulation
problem satisfy the SBCQ of the MPEC.

**Lemma 1:** For any convergent sequence $\{x^k, y^k\} \subseteq F$, there exists for each $k$ a multipliers
vector $\lambda^k \in M(x^k, y^k)$ and $\{\lambda^k\}$ is bound.

**Proof:** Clearly, if a vector $(x^k, y^k) \in F$, then a vector $x^k$ lies in $S(y^k)$ (or in the other word,
$x^k \rightarrow \text{sol} \left\{ \text{VI} \left( t(x^k, y^k), \Psi(y^k) \right) \right\}$ if and only if $x_k$ is an optimal solution of the following
convex program in variable $v$:

$$\min_v v^T \cdot t(x^k, y^k) \hspace{1cm} s.t. \ v \in \Psi(y^k)$$  \hspace{1cm} (14)

Recall that $t(x, y)$ is assumed to be strongly monotone with respect to $x$, hence $t(x, y)$ is
convex. Also, $g(x, y)$ which defines $\Psi(y)$ is assumed to be concave and the constraint
qualification of the linear independence of $\nabla_i g_\cdot (x, y), i \in I(x, y)$ is satisfied. Thus, this is a
well defined convex program (or more strongly linear program), it follows that its KKT
conditions (or the dual problem of this problem does has a solution), which are exactly those
given in (12), are well defined. Therefore for each $\{x^k, y^k\} \subseteq F$ there exists a multiplier
$\lambda^k \in M(x^k, y^k)$. Because $t(x, y)$ is bounded for all $(x, y) \in F$, the vector $\lambda$ is bounded.

**Theorem 1:** The problem $MPEC-VI$ is equivalent to the following optimisation problem:
**MPEC-KKT:**

\[
\begin{align*}
\max_y f(x, y) \\
\text{s.t. } y \in Y_{ad} \\
t(x, y) - \nabla_y g(x, y) \lambda &= 0, \\
g(x, y) &\geq 0, \lambda \geq 0, \lambda^T \cdot g(x, y) = 0
\end{align*}
\]  

(15)

**Proof:** This is the direct consequence of SBCQ.

The MPEC-KKT is a single level optimisation problem with complementarity constraints (CPs), i.e. \(0 \leq \lambda \perp g(x, y) \geq 0\). The CPs are very complicated and difficult to handle (see Falk and Liu, 1995). Our strategy will now be to solve a sequence of smooth, regular one-level problems that progressively approximate MPEC-KKT. The smoothing technique introduced in the next section is based on Facchinei, et al (1996) and Sumalee and Watling (2003).

### 3.2 Smoothing Formulation of Traffic Regulation Problem

Given a non-smooth CP, \(0 \leq \lambda \perp g(x, y) \geq 0\), the strategy adopted in this section is to reformulate this CP through the use of the merit function proposed by Kanzow (1996) which is in the form:

\[
\phi_\mu(a, b) = \sqrt{(a - b)^2 + 4\mu^2} - (a + b)
\]

(16)

The most important property of this function is as follows [Kanzow, 1996, Lemma 3.1]:

**Proposition 1:** For every \(\mu\) we have

\[
\phi_\mu(a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = \mu^2.
\]

(17)

If we let \(\mu = 0\) we have \(\phi_\mu(a, b) = 2 \min(a, b)\). From this, we can say \(\phi_\mu(a, b) = 0 \Leftrightarrow a \perp b\).

The reason for using the perturbation parameter (\(\mu\)) is to make \(\phi_\mu(a, b)\) continuously differentiable everywhere. To illustrate this we derive the Jacobians of \(\phi_\mu(a, b)\):

\[
\nabla \phi_\mu(a, b) = \begin{pmatrix}
\frac{\partial \phi_\mu(a, b)}{\partial a} \\
\frac{\partial \phi_\mu(a, b)}{\partial b}
\end{pmatrix} = \begin{pmatrix}
\frac{a-b}{\sqrt{(a-b)^2 + 4\mu^2}} - 1 \\
\frac{b-a}{\sqrt{(a-b)^2 + 4\mu^2}} - 1
\end{pmatrix}
\]

(18)

If \(\mu\) is reduced to zero, it can be shown that \(\phi_\mu(a, b)\) is non-differentiable when \(a = b\). However, for \(\mu \neq 0\), this function is everywhere differentiable. Furthermore, for every \((a, b)\), \(\lim_{\mu \to 0} \phi_\mu(a, b) = a \cdot b, a \geq 0, b \geq 0\). The function \(\phi_\mu(a, b)\) is therefore a smooth perturbation of the CP.
Now, let us introduce the set-to-set mapping operator $W_{\mu} : \mathbb{R}^{n+m+l} \rightarrow \mathbb{R}^{m+l}$,

$$W_{\mu}(w) \equiv W_{\mu}(x, y, z, \lambda) \equiv \begin{pmatrix} t(x, y) - \nabla_x g(x, y) \lambda \\ g(x, y) - z \\ \xi_{\mu}(z, \lambda) \end{pmatrix}$$ (19)

where $z \in \mathbb{R}^l$ is the slack variable used to transform the inequality constraints to equality constraints and

$$\xi_{\mu}(z, \lambda) \equiv \left( \varphi_{\mu}(z_1, \lambda_1), \ldots, \varphi_{\mu}(z_l, \lambda_l) \right)^T \in \mathbb{R}^l.$$ (20)

Then, for every $\mu \neq 0$, we can define the smoothing optimisation problem for the traffic regulation problem as follows:

**MPEC-MF**

$$\begin{align*}
\max_y & f(x, y) \\
\text{s.t.} & y \in Y_{ad} \\
& W_{\lambda}(w) = 0
\end{align*}$$ (21)

Again, for any $\mu \neq 0$, MPEC-MF is a smooth optimisation problem although MPEC-KKT is non-smooth in general. From the property $(a, b)$, $\lim_{\mu \to 0} \varphi_{\mu}(a, b) = a \perp b$, $a \geq 0, b \geq 0$, it is clear that MPEC-MF coincide with MPEC-KKT and thus MPEC-VI. We denote $F_{\mu} \subseteq \mathbb{R}^{n+m+l}$ the feasible set of problem MPEC-MF. Next, the most important task is to prove two propositions:

(i) for any $\mu \neq 0$, the problem MPEC-MF has a solution, or in the other word, the set $F_{\mu} \subseteq \mathbb{R}^{n+m+l}$ is not empty and bounded, and

(ii) Let $w^k = \left( x^k, y^k, z^k, \lambda^k \right)$ is a KKT point (if exists) of the problem MPEC-MF with $\mu^k \neq 0$, (note that $k$ is not the power but the indices) and let $w^* = \left( x^*, y^*, z^*, \lambda^* \right)$ be the solution of MPEC-KKT; we need to show that when $\mu^k \to 0$ $w^k_{\mu} \to w^*$ and $w^*$ is the solution of MPEC-KKT.

For brevity the proof for (i) is omitted. Those who are interested in the detail of the proof, consult Sumalee and Watling (2003). We turn our attention to (ii) which will be a very important proposition to verify the convergence of the solution algorithm proposed in the §4. We denote $d\left( Y_{ad} \times \mathbb{R}^{m+l} \right)$ as the Euclidean distance function to the set $Y_{ad} \times \mathbb{R}^{m+l}$.

**Proposition 2:** Let $w^k$ and $\mu^k \to 0$ be two sequences such that, for every $k$, $w^k$ is a KKT point for the problem MPEC-MF with $\mu^k \neq 0$ with multipliers $(\alpha^k, \gamma^k, \sigma^k)$ and for some vector $s^k \in \partial d\left( w^k \big| Y_{ad} \times \mathbb{R}^{m+l} \right)$, and such that $\mu^k \neq 0$ for all $k$. Suppose that \{w^k\} $\to w^*$, \{$(\alpha^k, \gamma^k, \sigma^k)$\} $\to (\alpha^*, \gamma^*, \sigma^*)$, and \{s^k\} $\to s^*$ then $w^* = \left( x^*, y^*, z^*, \lambda^* \right)$, with the multipliers $(\alpha^*, \gamma^*, \sigma^*)$ and with the vector $s^*$, satisfies the KKT conditions for the problem.
MPEC-MF with $\mu = 0$ which is equivalent to MPEC-KKT.

**Proof.** Based on the well-established theory of necessary optimality condition for Lipschitz program (Clarke, 1983), a necessary condition for a point $w^k = (x^k, y^k, z^k, \lambda^k)$ to be optimal for the problem MPEC-MF is that the point to be stationary, i.e. that $0 \neq (\alpha^k, \gamma^k, \sigma^k) \in \mathbb{R}^{m+l}$, and $s^k \in \partial d \left( w^k \mid Y_{ad} \times \mathbb{R}^{m+l} \right)$ exist such that

$$0 \in \nabla f(x^k, y^k) + \nabla L(x^k, y^k, \lambda^k) \alpha^k + \nabla (g(x^k, y^k) - z^k) \gamma^k + \partial \varphi_\mu (z^k, y^k) \cdot \sigma^k + M \left\| (\alpha^k, \gamma^k, \sigma^k) \right\| s^k$$

(22)

where $M$ is a Lipschitz constant for $(f, \xi)$ around $w$ and

$$L(x^k, y^k, \lambda^k) = t(x^k, y^k) - \nabla_x g(x^k, y^k) \lambda^k.$$  

(23)

These conditions are the Karush-Kuhn-Tucker (KKT) condition for the problem MPEC-MF. Therefore, from these conditions, in order to prove the **Proposition 2** we only need to show that:

$$0 \in \nabla f(x^*, y^*) + \nabla L(x^*, y^*, \lambda^*) \alpha^* + \nabla (g(x^*, y^*) - z^*) \gamma^* + \partial \varphi_\mu (z^*, y^*) \cdot \sigma^* + M \left\| (\alpha^*, \gamma^*, \sigma^*) \right\| s^*$$

(24)

with $s^* \in \partial d \left( w^* \mid Y_{ad} \times \mathbb{R}^{m+l} \right)$.

From the assumption of the upper semi-continuity of the sub-differential mapping, $s^* \in \partial d \left( w^* \mid Y_{ad} \times \mathbb{R}^{m+l} \right)$ and the continuity assumption, it is easy to see that if

$$\lim_{\mu \to \infty} \varphi_\mu \left( z_i^k, \lambda^k \right) \in a \cdot b \quad : i = 1, \ldots, l,$$

(25)

then (24) follows from (22) by the simple continuity argument. Thus, we only have to prove that (25) is true. Let $e$ be an identity matrix with the size of $(n+m+l+l)$ and $e_j$ denotes the $j$-th column of $e$. Recall that

$$\partial (z_i^k \cdot \lambda^k) = e_{n+m+i} \quad \text{if } z_i^k < \lambda_i^k$$

$$\partial (z_i^k \cdot \lambda^k) = e_{n+m+l+i} \quad \text{if } z_i^k > \lambda_i^k$$

$$\text{conv}(e_{n+m+i}, e_{n+m+l+i}) \quad \text{if } z_i^k = \lambda_i^k$$

(26)

where conv defines the convex combination of two vectors. On the other hand, from (18) we have

$$\nabla \varphi_\mu \left( z_i^k, \lambda^k \right) = \left( \frac{z_i^k - \lambda_i^k}{\sqrt{(z_i^k - \lambda_i^k)^2 + 4(\mu^k)^2}} - 1 \right) e_{n+m+i} + \left( \frac{\lambda_i^k - z_i^k}{\sqrt{(z_i^k - \lambda_i^k)^2 + 4(\mu^k)^2}} - 1 \right) e_{n+m+l+i}$$

(27)

If we pass the limit in (27) and comparing with (26), it verifies that the condition in (25) is
true and we complete the proof that (24) is true.

Before moving to the proposed solution algorithm, we will spend sometime to reformulate our traffic regulation problem in an equivalent form to MPEC-MF and to validate the assumptions used in constructing the equivalent perturbed version of MPEC-MF to its MPEC-KKT. The equivalent KKT condition for UE, compared to (12) for VI, is (see Patriksson, 1994: Theorem 2.1):

\[ x(t(x,y) + A^T \lambda) = 0, \]
\[ x \geq 0, \quad (t(x,y) + A^T \lambda) \geq 0. \]

(28)

Two key blanket assumptions made earlier for the proof of convergence of the perturbed MPEC-MF to MPEC-KKT is (i) the strong monotonicity of \( t(x,y) \) and (ii) the linear independence constraint qualification of matrix \( A \). The assumption (ii) can be fulfilled easily by reducing the matrix \( A \) to a full-rank matrix. Thus, this requires no proof.

The assumption (i) is rather complicated since the vector mapping function \( t(x,y) \) is defined on the space of multi-commodity flows which is the disaggregate space of the total link flow. Obviously, the strong monotonicity condition is naturally held for the delay functions with the aggregate link flows as variables. However, this is not the case for the disaggregate link flows variables. The condition provided by \( t(x,y) \) is only the monotonicity condition not strong monotonicity condition (replace the sign \( > \) by \( \geq \) in formula 4). The monotonicity condition only provides the necessity of the KKT condition for the solution of VI as proved earlier, but does not support the assumption required in the proposition 2.

Fortunately, this proposition is only required for the proof of the global convergence of MPEC-MF to MPEC-KKT. This means as \( \mu^k \to 0 \), the solution of the perturbed MPEC-MF approaches the solution of the corresponding MPEC-KKT problem. Although, this may not be true theoretically for the case of the disaggregate link flows due to the violation of the strong monotonicity condition of \( t(x,y) \), we propose the conjecture that the convergence of the solution in practice in terms of the aggregate link flows from the perturbed MPEC-MF to MPEC-KKT still exists. The reason supporting this conjecture is the fact that the aggregate link flow solution to VI is definitely unique according to Smith (1979). Thus, any aggregate multi-commodity link flow solutions from the VI will produce the same aggregate link flows. Then, by using the continuity property (or sensitivity of link flows) around its neighbourhood (see Tobin and Friesz, 1988), there exists a solution in terms of the aggregate link flows as \( \mu \) is perturbed. Hence, using the argument in proposition it is possible to get the condition that as \( \mu^k \to 0 \) the solution of the perturbed MPEC-MF approaches the solution of its original MPEC-KKT in terms of the aggregated link flows. From this assumption and proposition 2, we are now ready to establish and prove the convergence of our algorithm in the next section.

### 3.3 Solution Algorithm

The equivalence construction of the smooth perturbed MPEC-MF and the MPEC-KKT of the network capacity model in the previous section allows us to apply various fruitful non-linear optimisation algorithms available to the problem. In this section, the Sequential Quadratic Programming (SQP) approach is adopted to solve the perturbed MPEC-MF problem. The reason for using SQP is its special ability to handle significant nonlinearities in the constraints which are the case for MPEC-MF problem (see formula (16)). Also, it is appropriate for such a large-scale problem which is the nature of the multi-commodity flow problem.
First, we introduce the overall algorithm for solving \textit{MPEC-MF}. The strategy is to solve the number of smooth problems of the \textit{MPEC-MF} each of which is corresponding to a value of $\mu^k$ ($k$ denotes the outer iteration number). The initial value of $\mu^k$ will be set. Then, the algorithm will gradually decrease $\mu^k \to 0$. At each outer iteration, the SQP algorithm will be used to solve the perturbed \textit{MPEC-MF}($k$); again $k$ denotes the outer iteration number. The overall process of the algorithm can be summarised as follows:

Step 0: Let $\{\mu^k\}$ be a sequence of nonzero numbers with $\lim_{k \to \infty} \mu^k = 0$ (or simply $\mu^k \to 0$ as the outer iteration proceeds). Set $k := 1$ and choose an initial solution.
Step 1: Use SQP to find the solution of \textit{MPEC-MF}($k$).
Step 2: Set $k := k + 1$, update the value of $\mu^k$, and return to Step 1.

Recall the perturbed \textit{MPEC-MF} in (21). Without lost of generality, assume that we ignore the constraint sets on the design variables ($Y_{ad}$). Let $Q(w)$ denote the Jacobian matrix of the $W_{i,\lambda}(w)$. The first-order (KKT) conditions of (21) is as follows:

\[
\Gamma = \begin{bmatrix}
\nabla f(w) - Q(w)^T \cdot z \\
W_{i,\lambda}(w)
\end{bmatrix} = 0.
\]  

(29)

The Jacobian of (29) is given by:

\[
\begin{bmatrix}
\Upsilon(w) & -Q(w)^T \\
Q(w) & 0
\end{bmatrix},
\]  

(30)

where $\Upsilon(w)$ denotes the Hessian of the Lagrangian of (21). The SQP will iteratively solve the approximated quadratic programming sub-problem to determine the descent direction of the solution vector. Suppose that at the iterate $(w_i, z_i)$ we define the quadratic program:

\[
\max_{\mathbf{p}} \frac{1}{2} \mathbf{p}^T \Gamma \mathbf{p} + \nabla f_i^T \mathbf{p} \\
s.t. Q \mathbf{p} + W_{i,\lambda} = 0
\]  

(31)

Clearly, the solution of (31) in terms of $(\mathbf{p}_i, \mathbf{u}_i)$ must satisfy the following condition:

\[
\Gamma \mathbf{p}_i + \nabla f_i - Q_i^T \mathbf{u}_i = 0 \\
Q \mathbf{p}_i + W_{i,\lambda} = 0
\]  

(32)

Thus, the inner loop for solving this approximated quadratic programming sub-problem is as follows:

Step 0: Set $i := 0$ and choose the initial solution $(w_0, z_0)$;
Step 1: Evaluate $f_i$, $\nabla f_i$, $\Gamma_i$, $W_{i,\lambda}$, and $Q_i$;
Step 2: Solve (31) to obtain $\mathbf{p}_i$ and $\mathbf{u}_i$;
Step 3: Set $w_{i+1} \leftarrow w_i + \mathbf{p}_i$ and $z_{i+1} \leftarrow \mathbf{u}_i$;
Step 4: If convergence criteria satisfied, STOP, otherwise return to Step 1.
4. NUMERICAL EXAMPLE

4.1 Example Network

To show the efficiency of the proposed algorithm, a numerical experiment on a small sample network is shown here. The network consists of 4 nodes and 10 links, as is described in Figure 2. OD traffic volumes are shown in Table 1, together with regulation policies. Note that if all links are not degraded, circular links reach to their capacities when OD traffic volume in Table 1 flows onto the network, meaning that if we assume that OD pattern is fixed, the sum of OD traffic volume in Table 1 is the maximum total traffic volume that the network can afford. A link performance function is set to be linear; \( t_a = t_0 + 5(v_a/c_a) \), where \( t_0 \) is a travel time in free flow condition, \( v_a \) is a link traffic volume, and \( c_a \) is a link capacity.

![Figure 2. Example Network](image)

**Table 1. OD Traffic Volume**

<table>
<thead>
<tr>
<th>OD</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>400</td>
<td>400</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>400</td>
<td>300</td>
<td>400</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>300</td>
<td>400</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>200</td>
<td>400</td>
<td>400</td>
<td>400</td>
</tr>
</tbody>
</table>

**Figure 3. Calculation Result (link failure probability=0.8)**

4.2 Example of Calculation Result

A calculation result in the case some links are degraded with link failure probability of 0.8 is shown as an example. Figure 3(a) illustrates the link traffic conditions and Figure 3(b) represents the estimates of destination-specific link traffic volumes. First of all, from link conditions, for bold links connecting nodes 2 and 4, 4 and 3, the link traffic volumes are equal to their capacities. Therefore, in this example, these two are the links determining the maximum network capacity. From Figure 3(b), for OD pair (4, 1), there are two paths (4-2-1 and 4-3-1), and their travel times are equal (9.62+8.53=8.15+10.0=18.15). This suggests that proposed solution algorithm reaches to the user equilibrium condition with maximum network capacity.
4.3 Capacity Reliability Measures Considering Traffic Regulation

By using the proposed $MPEC-MF$, we can calculate the network reliability measures considering traffic regulation. The capacity reliability proposed by Chen et al. (1999) is defined as,

$$R(\mu_r) = \text{Pr}(\mu \geq \mu_r)$$

and is calculated by Monte Carlo simulation method. For each degraded sample network, maximum OD multiplier $\mu$ is calculated by solving the problem below.

$$\max \mu \quad \text{s.t.} \quad v_a(\mu, \text{OD}) \leq c_a, \ \forall a \in L$$

In this study, the similar methodology is adopted, although the calculation of the capacity reliability is modified. In this numerical example, we assume that link degradation occurs independently, and maximum degradation of link capacity is 0.5. The Monte Carlo procedure is summarised as follows.

Step 0 : Set sample number $s := 1$;
Step 1 : Generate a uniform random number $U_{a1}$ ranging from 0 to 1 for each link;
Step 2 : if $U_{a1} > d$ (link failure probability), generate another uniform random number $U_{a2}$, and reduce the capacity of link $a$ as $(1 - 0.5U_{a2})c_a$.
Step 3 : Solve $MPEC-MF$ to obtain $O_s^*$, maximum traffic volume accepted on the degraded network of $s$.
Step 4 : If $s < s_{\text{max}}$, $s := s + 1$ and go to Step 1. Otherwise STOP.

From $O_s^*$, capacity network measures considering traffic regulation with threshold of $\mu_r$ can be calculated as follows.

$$R(\mu_r) = \text{Pr}(\mu \geq \mu_r) = \sum_{s=1}^{s_{\text{max}}} \gamma_{\mu_r} / s_{\text{max}}.$$  \hspace{1cm} (35)

Where, $\gamma_{\mu_r} = 1$ if $O_s^*/O^0 \geq \mu_r$, $O^0$ is maximum traffic volume on basal case. $R(\mu_r)$ is calculated by changing $d$ and $\mu_r$. To compare with the difference in our capacity reliability measure with the original one, $R(\mu_r)$ proposed by Chen et al. (1999) is also calculated. In this example, $s_{\text{max}}$ is set to be 50, and $d$ is changed from 0 to 1 by the interval of 0.2. The result of calculation is shown in Figure 4.

![Figure 4. Capacity Reliability](image_url)

From figures, the values of capacity reliability measure are different, indicating that the
measure to show the anti-hazardous ability of the network may differ when we consider the traffic regulation. This suggests the importance in considering the traffic regulation to evaluate the network. In this example, the capacity reliability measure considering traffic regulation is larger than the original one. Judging from the model structure, it can be said that our model has one more degree of freedom because we have two multipliers, $\theta$ and $\phi$. This may be the reason for this result. However, in our model, we have fixed demand that must be assigned (the third term in objective function (1)). In this example, in order not to have a situation that the network can not accept these fixed demand, the capacity of links connecting nodes 2 and 3 are set to be always larger than the demand without regulation. However, in reality, it may well happen that the network can not afford the traffic demand that is not regulated. In our model, it means that we do not have feasible solution set, and we can not obtain an efficient entry permission rates.

5. CONCLUSIONS

This paper proposed the methodology to evaluate the anti-hazardous ability of the network by the concept of network capacity reliability. The model to calculate the entry permission rates for two-stage area regulation is formulated with MPEC, which is one of the most challenging problems in optimisation. In order to utilise the proposed model to network reliability analysis, more efficient algorithm to solve MPEC is needed. This paper reformulates the original bi-level optimisation problem into a smooth single level optimisation problem by utilising the idea of merit function. Mathematical equivalence of the reformulated problem is shown. As transformed problem is a smooth single level optimisation problem, any solution algorithms solving nonlinear optimisation problem can be applied. In this paper, sequential quadratic programming approach is used. The application of the proposed algorithm into a small sample network suggests that it converges to the optimal solution of the original model. It was also found that the values of network capacity measure considering traffic regulation are different from the original ones, suggesting that the measures of the anti-hazardous ability of the network may differ when we consider the traffic regulation. In this study, the example of small network is shown as an example. It is preferable to apply the model onto real-size network. The current problem for this is the definitions of the variables. As the traffic volumes are defined as destination-specific link traffic volume, the number of unknown variables and constraints increases enormously when the size of the network increases. Therefore, some relaxation of the problem to reduce the size of problem is needed. Another discussion should be made whether the assumption of user equilibrium condition is feasible to evaluate the emergent situation of the road network.

ACKNOWLEDGEMENT

The second author would like to thank the UK Department for Transport for the funding under the New Horizon research programme.

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