ANALYTICAL SOLUTIONS FOR PREDICTING TRAIN COASTING DYNAMICS

Jyh-Cherng Jong
Ph. D., P.E., Research Scientist
Civil and Information Technology Research Center
Sinotech Engineering Consultants, Inc.
171 Nanking E. RD. SEC. 5
Taipei 105, Taiwan
Fax: +886-2-27692131 Ext. 20998
E-mail: jcjong@sinotech.org.tw

Abstract: A primary study of railway operations is the calculation of train dynamics under different operation regimes, including powering, constant speed, coasting, and braking. Due to the reason that forces affecting train movement are complicated, it is hard to calculate train dynamics analytically and thus, numerical methods are widely used in railway industries to simulate train movement. Unlike previous studies, this paper tries to derive analytical solutions for predicting train dynamics under coasting operation. The analytical model provides a quick way to analyze train coasting behaviors. Since the solutions are exact, they can be used to validate the results from numerical methods and to facilitate the development of train performance simulators.

Key Words: Train Dynamics, Coasting, Analytical Solutions, Simulation

1. INTRODUCTION

One of the basic planning tasks for railway operation is to simulate the movement of a train along a rail line. Under different circumstances, the train may alternate operation modes to move from origin to destination, including powering, coasting, constant speed, and braking. During the course of the movement, the train dynamics (i.e., the changes of train position, velocity, and acceleration with respect to the change of time) are governed by Newton's Second Law of Motion, which is the second-order ordinary differential equation (ODE) of distance with respect to time. An intuitive approach to solve the ODE is to integrate it over time. Unfortunately, forces that influence train movement are not explicit functions of time. On the contrary, most of them are velocity dependent. Consequently, railway industries employ numerical methods to approximate the solutions. The algorithms all assume that the net force acting on the train is constant over a short section and then use iterative computational cycles based on time, distance, or velocity increments to calculate train dynamics (Howard, 1983). For example, AREMA (1999) and Andrews (1986) suggest a velocity increment worksheet for such calculations. Kikuchi (1991) uses distance increment method to simulate train travel on a rail transit line. Capillas (1987), Goodman (1987), and Uher (1987) adopt a time increment approach. A similar time-based scheme can also be found in Rakha (2001) for predicting maximum truck acceleration performance on grades.

The algorithms mentioned above can be carried out by hand calculations or computer programs (i.e., train performance simulator). The velocity increment worksheet has been widely used in railway industries for more than 50 years and only been undertaken by experienced staff. Recently, with the prevalence of computers, the algorithms are computerized, mostly based on time increment. Note that train dynamics resulted from numerical methods are not precise. The accuracy depends on the magnitude of the increment. For powering and braking operations, numerical approaches may be inevitable since tractive effort and braking force may not have precise function forms. However, for coasting operation, it would be possible to derive analytical solutions. This paper presents such a model for estimating the theoretical values of train coasting dynamics. The analytical solutions can be used to validate the results from a train performance simulator and to analyze the errors of different numerical methods.

2. TRAIN DYNAMIC MODEL FOR COASTING OPERATIONS

The forces acting on a moving train may include tractive effort, train running resistance, alignment resistance, and braking force. Tractive effort provides the propulsion to overcome resistances and to accelerate the train. Running resistance is the force opposing the movement of the train. Alignment resistance is composed of grade resistance and curve resistance. Both are due to the geometry of railway alignment. Braking force is used to decelerate the train and to bring it to full stop. For coasting operation, the tractive effort and braking force of the train are cut off. Therefore, only running and alignment resistances are in effect. Each of them is discussed in the following subsections. Finally, the equation of motion that governs the behavior of coasting operation is presented.

2.1 Running Resistance

Train running resistance is caused by many factors and is very complex. The analysis of running resistance is usually decomposed into individual components. Vuchic (1981) classified it into two main categories: basic resistance and air resistance, as shown in Figure 1. The first one is purely mechanical and can be further decomposed into rolling resistance and way resistance. However, they are rarely analyzed separately since individual resistances usually cannot be measured precisely. Experimental studies indicate that some portion of basic resistance is constant at all speeds, while the other is proportional to train velocity. On the other hand, air resistance depends on the square of the relative velocity of train to air, which is equal to train velocity in still air.

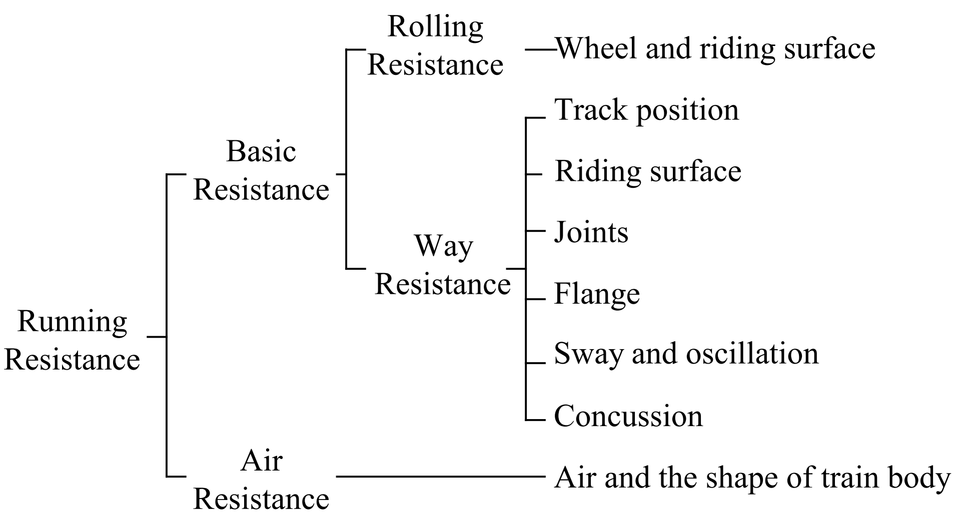

Figure 1 The Composition of Train Running Resistance

Running resistance is obtained by summing all individual elements together and can be expressed as the following quadratic form of train velocity:

$$R_r = A + BV + CV^2 \tag{1}$$

where: R_r = running resistance (N)
A, B, C = positive constants for a specific train (usually obtained from field tests or provided by train manufactories)
V = train velocity (km/h)

Since most of train resistances vary with trainload, it is convenient to express them through resistance coefficients, which are defined as the resistance in N per kN weight of the train. Thus, equation (1) can be further rewritten as

$$R_r = r_r W = (a + bV + cV^2)W \tag{2}$$

where: r_r = coefficient of running resistance (‰)
W = train weight (kN)
a, b, c = positive constants; $a = A/W$, $b = B/W$, and $c = C/W$

2.2 Alignment Resistance

Alignment resistance includes grade resistance and curve resistance. The former can be computed exactly, whereas the latter is obtained empirically.

1. Grade Resistance

Figure 2 shows that the gravity force of a train on a grade can be resolved into two components: the force along the slope and the force normal to the incline. The former creates a resistance to the movement of the train and is defined as grade resistance. From simple trigonometry, it can be shown that grade resistance is

$$R_g = 1000 W \sin\theta \tag{3}$$

where: R_g = grade resistance (N)
θ = slope (°)

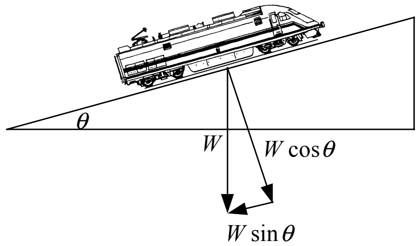

Figure 2 The Illustration of Grade Resistance

For railway alignments, θ is usually very small and for such values, $\sin\theta \approx \tan\theta$. Let r_g be the coefficient of grade resistance (‰). Then $r_g = 1000\tan\theta$ and equation (3) may be rewritten as

$$R_g = r_g W \tag{4}$$

Note that grade resistance is only a general term. It is not necessarily a resistance to train motion. The actual effect depends on the sign of θ. For $\theta > 0$, which indicates an incline, grade resistance is in the opposite direction of train movement. On the other hand, for $\theta < 0$, which represents a decline, grade resistance is in the same direction of train movement and can contribute to train acceleration. If, however, $\theta = 0$ (i.e., a level alignment), then grade resistance is neither an accelerating force nor a decelerating force for the train.

2. Curve Resistance

When a train runs in a curve, its instantaneous moving direction is tangent to the curve. Since rail tracks force the train to travel along the curve, the centrifugal force acting on the train and the friction between wheels and tracks produce extra resistance to train movement. The additional resistance is called curve resistance, which can be expressed as

$$R_c = r_c W \tag{5}$$

where: R_c = curve resistance (N)
r_c = coefficient of curve resistance (‰)

The coefficient of curve resistance r_c depends on the friction coefficient, the gauge, the distance between axles in vehicles, and the radius of the curve. It usually increases as the increases of gauge, axle distance, and friction coefficient, but decreases as radius increases. For a specific rail system, the above factors are considered to be known data except radius, which varies with alignment geometry. Hence, r_c can be expressed as a constant K multiplied by the reciprocal of the radius γ (m), i.e.,

$$r_c = \frac{K}{\gamma} \tag{6}$$

The typical value of K in equation (6) ranges from 500 to 800, depending on rail systems. Some of them are listed in Table 1. Note that different K values do not make big differences in computing r_c. For example, when the radius of a curve is 300 m (which is very small for rail systems), the difference between the resulting r_c for $K = 500$ and $K = 800$ is only 1‰. When the radius is 3000 m, the difference is even as small as 0.1‰!

3. Equivalent Grade

Since r_g and r_c are independent of train velocity and both are in ‰, it is convenient to add them together. Let r_e denote the sum of r_g and r_c, i.e.,

$$r_e = r_g + r_c \tag{7}$$

Then r_e represents the coefficient of total alignment resistance and is called "Equivalent Grade" (Hay, 1982). A train travels on a grade of r_g with a curve resistance coefficient r_c is equivalent to traveling on a grade of $r_e = r_g + r_c$.

Table 1 The Typical Values of K in Computing Curve Resistance Coefficient

Rail System	The Value of K
SNCF	800
Japan	800
AREMA	700
China	700
Department of Rapid Transit Systems, Taipei City Government	700
Taiwan Railway Administration (TRA)	600
IR Universalformel	516
RFFS Universalformel	505

2.3 Total Resistance

The total resistance to the motion of a train, denoted by R_t (N), is the sum of running resistance and alignment resistance

$$R_t = R_r + R_g + R_c = (r_r + r_g + r_c)W \tag{8}$$

Let r_t be the coefficient of total resistance (‰). Then

$$r_t = r_r + r_g + r_c = a + bV + cV^2 + r_e \tag{9}$$

Since r_e is independent of V, it can be added to the constant term a. Let $\alpha = a + r_e$. Then r_t can be further expressed as follows:

$$r_t = \alpha + bV + cV^2 \tag{10}$$

Note that r_t only makes sense for $V \geq 0$ because train speed cannot be negative. However, it might be interesting to investigate its mathematical properties for the entire range of V. Since $r_t''(V) = c > 0, \forall V$, r_t is convex and there exists

$$V^* = \frac{-b}{2c} < 0 \text{ (the root of } r_t'(V) = 0\text{)} \tag{11}$$

such that $r_t(V^*) < r_t(V), \forall V \neq V^*$ and the global minimum is

$$r_t(V^*) = \frac{4\alpha c - b^2}{4c} \tag{12}$$

For a level and straight alignment (i.e., $r_g = r_c = 0$), r_t is identical to r_r. In such a case, the constant α is usually much greater than b and c for rail trains. Therefore, $b^2 - 4\alpha c < 0$ and r_t does not have any real roots. For a specific $r_e < 0$, it might be possible that $b^2 - 4\alpha c = 0$ and r_t has exactly one real root equal to V^*. If $r_e << 0$ (i.e., a steep downgrade), then $b^2 - 4\alpha c > 0$ and the curve of total resistance coefficient crosses the abscissa at two distinct points. This means that r_t has two real roots and their values are given by $(-b \pm \sqrt{b^2 - 4\alpha c})/2c$. The smaller one $(-b - \sqrt{b^2 - 4\alpha c})/2c$ is always negative, but the bigger one $(-b + \sqrt{b^2 - 4\alpha c})/2c$ may be a positive number. In case that the root is greater than zero, it is defined as "Balancing Speed", a speed at which the net force acting on the train is equal to zero. Balancing speed plays an important role in train operation. If a train travels at a speed greater than balancing speed, the train will decelerate down to balancing speed and then maintain at that speed. On the other hand, if the train operates at a speed less than balancing speed, the train will accelerate until its speed reaches balancing speed.

The r_t curves in the three situations of $b^2 - 4\alpha c > 0$, $b^2 - 4\alpha c = 0$, and $b^2 - 4\alpha c < 0$ are well explained in Figure 3. The figure clearly shows that when a train travels on a steep decline, the total resistance may be negative. In such a case the total resistance is in the same direction of train movement and the train is able to accelerate without tractive effort (i.e., coasting results in train acceleration).

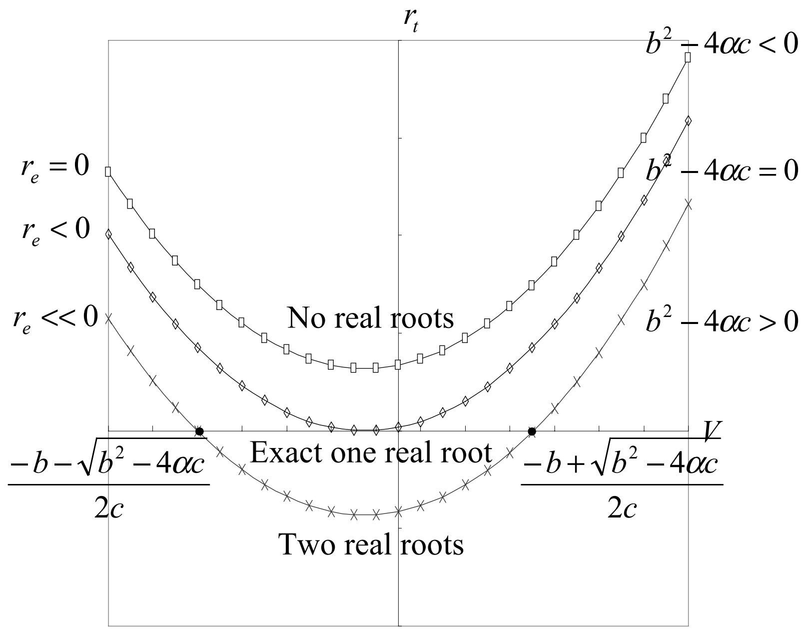

Figure 3 The effect of r_e on the Mathematical Properties of r_t Curve

2.4 Equation of Motion for Coasting Operation

Coasting is widely utilized in daily train operation for energy savings and to provide slack time for a late train to catch up its schedules by using full performance. When a train starts coasting operation, its tractive effort is turned off and the only force acting on the train is resistance. Let M be the static mass of the train in kg and assume that the train moves in the positive direction. Then Newton's Second Law of Motion states

$$-R_t = \rho M \frac{dv}{dt} \tag{13}$$

where v is train velocity in m/s and t is time in s. Since the rotating parts of a train, such as wheels, rollers and transmission gears may store energy gained from the force applying to the train, the parameter ρ ($\rho > 1$) is used to account for the rotating mass of the train and ρM

is called “Equivalent Mass” (Andrews, 1986). The typical value of ρ is in between 1.04 and 1.10 (Vuchic, 1981).

Plugging equations (8) into (13) and converting the unit of speed from m/s to km/h yield

$$-(\alpha + bV + cV^2)W = \frac{1000\rho W}{g}\frac{dV}{3.6dt} \tag{14}$$

Canceling out W on both sides of equation (14) and substituting 9.81 m/s^2 for gravity acceleration g lead to

$$-\frac{1}{28.32\rho}(\alpha + bV + cV^2) = \frac{dV}{dt} \tag{15}$$

Equation (15) will be used latter to derive the analytical solutions to train coasting dynamics.

3. ANALYTICAL SOLUTIONS TO TRAIN COASTING OPERATIONS

3.1 Fundamentals of The Solution Approach

The train dynamics under coasting operation are governed by equation (13). Since $v = ds/dt$, the equation of motion is the second-order ODE of s with respect to t. A direct method to solve such an ODE is to integrate it over t. But this does not work for equation (13). The difficulty arises from the fact that R_t is not an explicit function of t. It can be seen from equation (14) that R_t is velocity dependent. That is the main reason that previous studies adopt numerical methods to approximate the solutions.

To solve equation (13) analytically, the independent variable must be changed from t to v. Assume that a force $F(v)$ (N) acts on an object of mass m (kg). Rearranging the equation of motion and integrate it over v gives

$$t = \int dt = m\int \frac{1}{F(v)}dv \tag{16}$$

By definition, $ds = vdt = vdv/F$ and we obtain

$$s = \int ds = m\int \frac{v}{F(v)}dv \tag{17}$$

Both equations (16) and (17) take v as the independent variable for the integration. Therefore, they can be used to solve train running time and running distance, given train velocities.

3.2 Analytical Solutions to Train Running Time

Rewriting equation (15) in the form of equation (16) yields

$$t = -28.32\rho \int \frac{1}{\alpha + bV + cV^2} dV \tag{18}$$

The results of the integration depend on the sign of $b^2 - 4\alpha c$. If $b^2 - 4\alpha c > 0$, then equation (18) can be rewritten as

$$t = \frac{-28.32\rho}{c} \int \frac{1}{\left(V + \frac{b}{2c}\right)^2 - \left(\frac{\sqrt{b^2-4\alpha c}}{2c}\right)^2} dV \tag{19}$$

By decomposing the function inside the integration

$$\frac{1}{\left(V + \frac{b}{2c}\right)^2 - \left(\frac{\sqrt{b^2-4\alpha c}}{2c}\right)^2} = \frac{1}{\frac{\sqrt{b^2-4\alpha c}}{c}} \left(\frac{1}{V + \frac{b}{2c} - \frac{\sqrt{b^2-4\alpha c}}{2c}} - \frac{1}{V + \frac{b}{2c} + \frac{\sqrt{b^2-4\alpha c}}{2c}} \right) \tag{20}$$

It is easy to verify that

$$t = \frac{-28.32\rho}{\sqrt{b^2 - 4\alpha c}} \ln \left| \frac{2cV + b - \sqrt{b^2 - 4\alpha c}}{2cV + b + \sqrt{b^2 - 4\alpha c}} \right| + t_0 \tag{21}$$

If $b^2 - 4\alpha c = 0$, then $\alpha + bV + cV^2 = c(V + b/2c)^2$ and the result of (18) is

$$t = \frac{-28.32\rho}{c} \int \frac{1}{\left(V + \frac{b}{2c}\right)^2} d\left(V + \frac{b}{2c}\right) = \frac{56.64\rho}{2cV + b} + t_0 \tag{22}$$

If $b^2 - 4\alpha c < 0$, then equation (18) is rewritten as

$$t = \frac{-28.32\rho}{c} \int \frac{1}{\left(V + \frac{b}{2c}\right)^2 + \left(\frac{\sqrt{4\alpha c - b^2}}{2c}\right)^2} dV \tag{23}$$

and the result of the integration is

$$t = \frac{-56.64\rho}{\sqrt{4\alpha c - b^2}} \tan^{-1} \frac{2cV + b}{\sqrt{4\alpha c - b^2}} + t_0 \tag{24}$$

Equations (21), (22), and (24) express train running time as functions of train velocity. Hence, they can be used to forecast train running time at different speeds.

3.3 Analytical Solutions to Train Running Distance

Following equations (15) and (17), we obtain

$$s = -7.87\rho \int \frac{V}{\alpha + bV + cV^2} dV \tag{25}$$

Using the technique of integration by part, equation (25) can be rewritten as

$$
\begin{aligned}
s &= \frac{3.93\rho b}{c}\int \frac{1}{\alpha + bV + cV^2} dV - \frac{3.93\rho}{c}\ln\left|\alpha + bV + cV^2\right| \\
&= \frac{-b}{7.2c}t - \frac{3.93\rho}{c}\ln\left|\alpha + bV + cV^2\right| + s_0
\end{aligned}
\tag{26}
$$

Substituting equations (21), (22), and (24) for t, we obtain the following results:

$$
s = \begin{cases}
\dfrac{3.93\rho}{c}\left(\dfrac{b}{\sqrt{b^2 - 4\alpha c}}\ln\left|\dfrac{2cV + b - \sqrt{b^2 - 4\alpha c}}{2cV + b + \sqrt{b^2 - 4\alpha c}}\right| - \ln\left|\alpha + bV + cV^2\right|\right) + s_0, & \text{if } b^2 - 4\alpha c > 0 \\
\dfrac{3.93\rho}{c}\left(\dfrac{-2b}{2cV + b} - \ln\left|\alpha + bV + cV^2\right|\right) + s_0, & \text{if } b^2 - 4\alpha c = 0 \\
\dfrac{3.93\rho}{c}\left(\dfrac{2b}{\sqrt{4\alpha c - b^2}}\tan^{-1}\dfrac{2cV + b}{\sqrt{4\alpha c - b^2}} - \ln\left|\alpha + bV + cV^2\right|\right) + s_0, & \text{if } b^2 - 4\alpha c < 0
\end{cases}
\tag{27}
$$

The above equation describes running distance in functions of velocity and thus, can be used to calculate train position at different speeds.

4. AN EXAMPLE

A push-pull train of Taiwan Railway Administration (TRA) is employed here to demonstrate the analytical model developed in the previous section for predicting coasting dynamics on different grades. The train is the primary intercity passenger train of TRA that serves the west coast of Taiwan. It is made up with two electric locomotives of E1000 type on each end and several passenger cars in between them. The pair of the locomotives is 120 ton in weight (1176.84 kN) and each passenger car is 35 ton in weight (343.2 kN). The speed limit of the train is 130 km/h and the equivalent mass ratio ρ is 1.06.

Assume that a pair of locomotives hauls 12 passenger cars to form a train. Then the total weight of the train is 5295.8 kN and the coefficient of running resistance is

$$
r_r = 1.3467 + 0.00897V + 0.000303V^2 \tag{28}
$$

At the beginning, it is assumed that train position and accumulative running time are zero. In order to analyze the effects of grades on coasting behaviors, train dynamics on different grades are calculated, where the initial velocities of the train are set to 130 km/h or 30 km/h.

Using the analytical model presented in the previous section, we are able to calculate train position and running time with respect to velocity. Figure 4 plots the relation between velocity and running time for grades from -20‰ to 20‰ with an increment of 5‰. It shows that on grades below -5‰, coasting results in acceleration for initial velocity = 30 km/h and the acceleration rate increases as the grade decreases (i.e., a steeper downgrade). On the other hand, for initial velocity = 130 km/h, coasting leads to deceleration on grades above -5‰, and the deceleration rate (i.e., negative acceleration rate) increases as the grades increases. A similar relation between train velocity and position is also given in Figure 5, where the velocity curve to distance is relatively nonlinear as compared with that to running time.

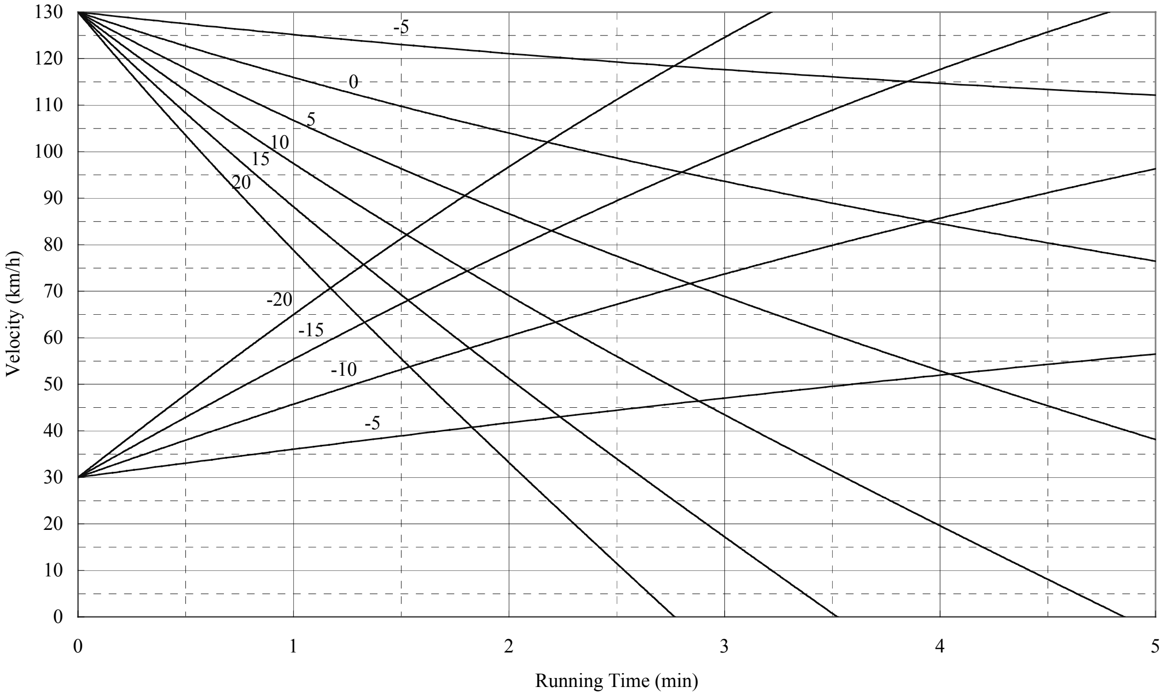

Figure 4 Velocity Curve to Running Time for Different Grades

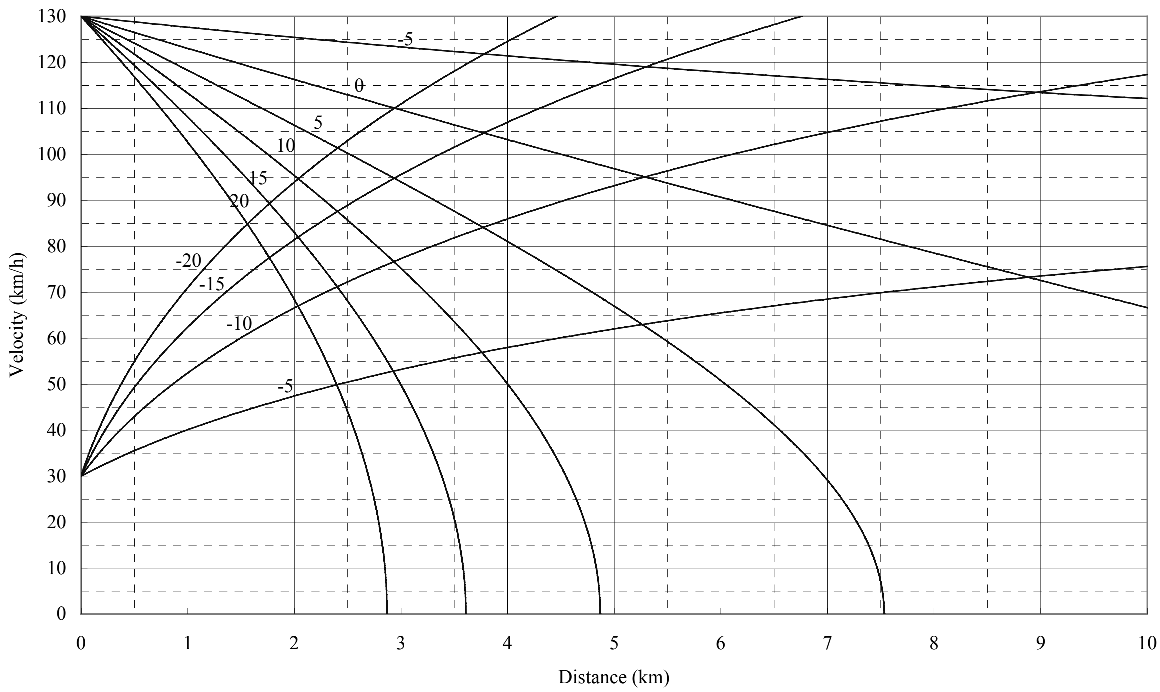

Figure 5 Velocity Curve to Train Position for Different Grades

Both Figure 4 and Figure 5 indicate that on grade of -5‰, the train decelerates if its initial velocity is 130 km/h, but accelerates in case of 30 km/h. An interesting question would be what is the balancing speed on grade of -5‰. To answer this question, we may simulate train movement over a sufficient long time or distance. Suppose that the train coasts from different initial velocities. Using the analytical model developed in section 3, we can calculate train dynamics for each initial condition. The velocity curves with respect to running time and running distance are plotted in Figure 6 and Figure 7, respectively. It can be seen that each curve converges to 96 km/h, which represents the balancing speed on grade of -5‰. Any initial velocity differs from it will result in acceleration or deceleration until train velocity reaches 96 km/h.

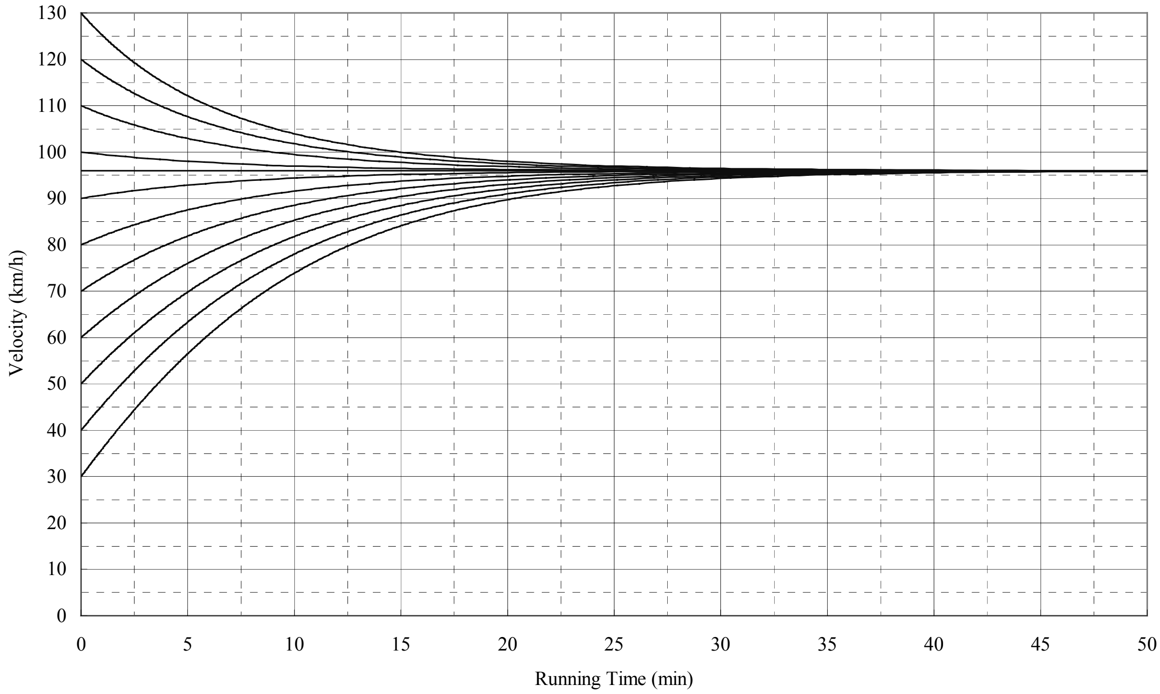

Figure 6 Velocity Curves to Running Time for Different Initial Conditions

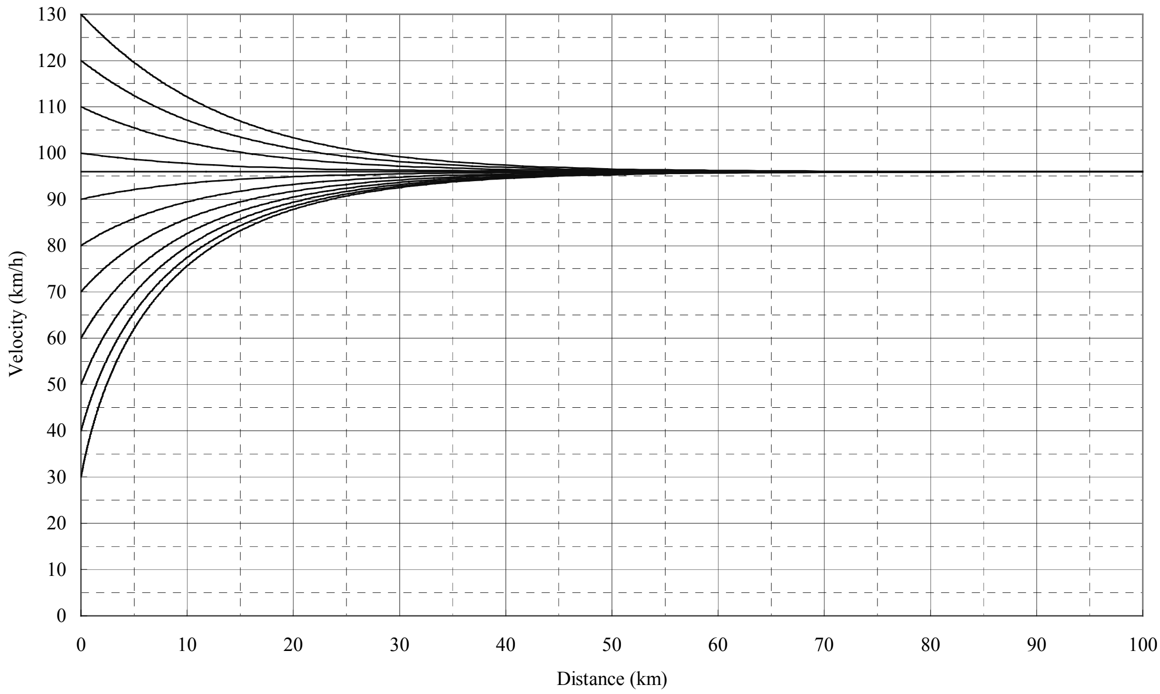

Figure 7 Velocity Curves to Train Position for Different Initial Conditions

Another approach to calculate balancing speed for coasting operation is to find the positive root of the total resistance coefficient. Figure 8 depicts the coefficient of total resistance on grade of -5‰. It can be shown that the positive root $(-b+\sqrt{b^2-4\alpha c})/2c$ is also 96 km/h. The result is consistent with Figure 6 and Figure 7.

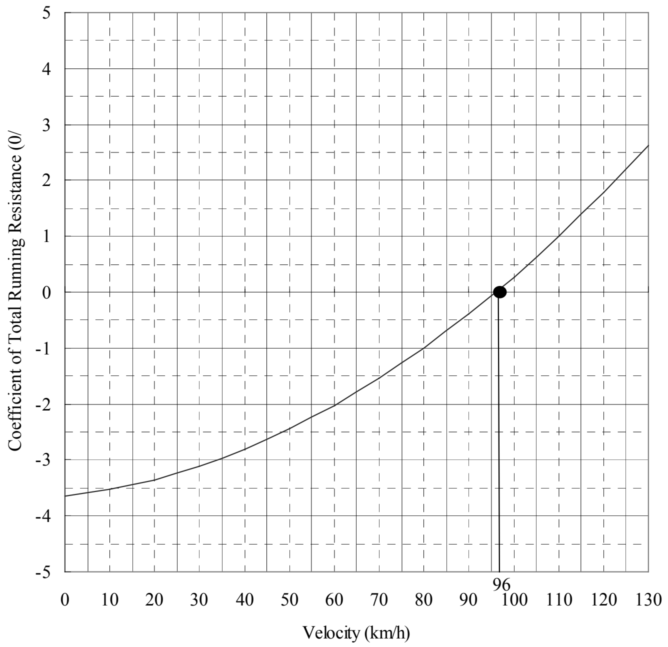

Figure 8 The Coefficient of Total Resistance on Grade of -5‰

5. CONCULSION REMARKS

Traditional methods for analyzing train movement use iterative computation cycles based on the increments of time, distance or velocity. These numerical approaches require heavy computations that are tedious for hand calculations. Although all algorithms can be computerized, a robust program still takes time to develop and the results are only approximate. Therefore, this paper develops an analytical model to facilitate the analysis. The model provides a convenient way to forecast train coasting dynamics. The solutions are precise mathematically and thus, can be used to validate the results from numerical programs. In addition, the analytical solutions can be extended to estimate train dynamics under powering or braking operations provided that quadratic functions of velocity approximate tractive effort and braking force with acceptable accuracy.

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