Abstract: We consider a city with several facilities that compete with each other for customers of different classes. The transportation system of the city is a highly dense road network. The customers are distributed continuously in the city, and each chooses a facility based on his or her transportation cost to the facility and the in-situ cost at the facility. We assume that different classes of users perceive costs differently, and hence are subject to different transportation cost functions and market externalities. An equivalent mathematical model is formulated and proved to satisfy the user equilibrium conditions. The resultant problem is solved by the finite element method. Numerical examples for both cases of fixed and elastic demand are given to illustrate the effectiveness of the proposed methodology.

Key Words: continuous model, traffic equilibrium, multiple user classes, finite element method, Newtonian algorithm

1. INTRODUCTION

For the equilibrium prediction and solution algorithm of the flow in a transportation system, there are, in general, two approaches in the literature: discrete modeling and continuous modeling. In the discrete approach, different zones are treated as centroids, and roads are considered as nodes and links that join these centroids to form a discrete network. In the continuous approach, the major assumption is that the variation in the nearby areas is relatively small when compared to the entire system. Thus, the characteristics of the transportation system, such as flow and cost, can be represented by smooth mathematical functions (Vaughan, 1987).

Traditionally, traffic equilibrium problems are studied with the discrete modeling approach. However, much attention has recently been paid to the continuous modeling approach to cope with the deficiencies of the discrete approach. First, for the strategic planning of a very dense transportation system, the discrete approach involves a large number of unknowns that are generally difficult to solve in an efficient manner. Second, it has to assume that the demand is concentrated at a centroid, which is unrealistic because the demand is distributed over the region. Thus, the continuous approach can model these aspects in a more effective manner.
A sizeable literature focuses on the continuous modeling approach. In some of the early studies, such as those of Buckley (1979), Dafermos (1980), D’Este (1987), and Wong (1994), the continuous models for cities were formulated with specific shapes (such as circular or rectangular), which was not adequate to model real life situations. Some other studies considered an arbitrary shape for the city configuration (Beckmann, 1952; Zitron, 1974; Williams and Ortuzar, 1976). Sasaki et al. (1990) further extended the idea of continuous modeling for a city with arbitrary shape and used the finite element method to solve the resultant problems. Wong et al. (1998) developed a more robust finite element procedure to solve the problem. Based on this framework, a number of extensions were made to apply the continuous modeling approach to traffic equilibrium problems, which include the estimation of market areas (Wong and Yang, 1999; Yang and Wong, 2000), and simultaneous discrete and continuous modeling (Yang et al., 1994; Wong et al., 2002a). For practical applications, Ho et al. (2002) has successfully applied the continuum model for the airport coordination problem in the Pearl River Delta region in China.

All of the above continuum models for arbitrary city shapes were for single user classes, which is unrealistic because, in practical terms, networks are shared by different types of users whose perceived travel cost and congestion externality can be very different. A single user class cannot closely model the real situation. Hence, it is necessary to consider multiple user classes in the continuous modeling method (Wong, 1998; Wong and Sun, 2001; Wong et al., 2002b). The present research is inspired by that of Vliet et al. (1986), which was developed for the discrete network that models how different classes of users interact with each other.

We develop a continuous equilibrium model with multiple user classes, in which each class of users has a distinct cost function. The cases of elastic demand and market externality are also considered. In Section 2, the modeled city with competitive facilities, together with the customer’s behavioral assumptions, are introduced. The formulation of a multi-class user equilibrium problem as an equivalent mathematical program is given in Section 3. Section 4 derives a finite element solution algorithm to solve the resultant problem. Section 5 presents two numerical examples to illustrate the effectiveness of the proposed methodology.

2. THE MODELLED CITY

Consider a city in a two-dimensional plane as illustrated in Figure 1, in which the road network is approximated as a continuum. In this city, there are $N$ facilities (such as shopping centers, industrial centers, or public transportation terminals) competing with each other to attract customers of $M$ different classes whose home locations are spread over the whole city. The dense transportation network is approximated as a continuum. The detailed treatment of the relationship and conversion between the dense discrete network and the approximated continuum can be found in Sasaki et al. (1990). The customers travel from their home locations through the continuum to patronise any one of the facilities. Denote the city as $\Omega$, the boundary of the study region as $\Gamma$, and the location of the $n$th facility as $O_n$.

The distribution of demand for class $m$ customers over the city, $\Omega$, is assumed to be continuous and represented by a non-negative, heterogeneous density function $q_m(x,y)$, where $q_m(x,y)$ is the total demand per unit area, from location $(x,y)\in\Omega$, to patronize any one of the facilities. For the case of elastic demand, the traffic demand of a particular class of customers
at any point within the city is directly related to their total transportation cost from that point
to their chosen facility plus the in-situ cost at the facility. The demand function can be
specified as
\[ q_m(x,y) = D_m(x,y,u_m(x,y)) \]  
where \( u_m(x,y) \) is the total transportation and in-situ cost for class \( m \) customers from the point
\((x,y)\) in the city to the chosen facility, and the demand function \( D_m(.) \) is assumed to be a monotonic
decreasing function with respect to the total cost at that point. Denote \( q = (q_m(x,y), m = 1,2,\ldots,M) \).

The transportation cost for each class of customers in the
region is assumed to be dependent on the total local flow intensity and road configuration, but
not upon direction of any class of users (an isotopic case),

\[ c_m(x,y,f) = a_m(x,y) + \sum_{l=1}^{M} b_{ml}(x,y)|f_l(x,y)| \]  
where \( f = (f_m(x,y), m = 1,2,\ldots,M) \), \( c_m(x,y,f) \) is the transportation cost of class \( m \) customers
to travel a unit distance at co-ordinate \((x,y)\) in the city, \( a_m(x,y) \) and \( b_{ml}(x,y) \) are strictly positive scalar
functions, \( f_m(x,y) = (f_{mx}(x,y), f_{my}(x,y)) \) is a vector that represents the flow state of
customer class \( m \) in the city, \( f_{mx}(x,y) \) and \( f_{my}(x,y) \) are respectively the flow flux in the \( x \)
and \( y \) directions, and

\[ |f_m(x,y)| = \sqrt{f_{mx}(x,y)^2 + f_{my}(x,y)^2} \]  
is the norm of the flow vector for class \( m \) customers at \((x,y)\). For the cost function, the first
term represents the free flow (zero-flow) transportation cost per unit distance, and the second
term reflects the effects of congestion. The explicit dependence of the cost function \( c_m(x,y,f) \)
on location \((x,y)\) reflects the possibility that the density and capacity of the transportation
system may vary from point to point. Here, we consider a symmetric case where \( b_{ml} = b_{lm} \),
\( \forall m,l \).

Inside the domain of the city \( \Omega \), the flow vector and customers’ demand must satisfy the flow
conservation condition, and this can be specified by

\[ \nabla f_m + q_m = 0, \quad \forall (x,y) \in \Omega, \forall m = 1,\ldots,M \]  
Assuming that no flow crosses the boundary of the city, we have

\[ f_m = 0, \quad \forall (x,y) \in \Gamma, \forall m = 1,\ldots,M \]  
Note that it is straightforward to extend the model to deal with the case of \( f_m \cdot n = g_m(x,y) \)
on the boundary \( \Gamma \), where \( n \) is the normal vector on the boundary and \( g_m \) is a function that
represents the given demand distribution of class \( m \) customers entering or leaving the city
through the boundary. To avoid singularity at the facilities, we assume that each of the
facilities are of a finite size enclosed by a clockwise boundary segment \( \Gamma cn, n = 1, 2,\ldots,N \).
Denote $\Omega_{mn}$ as the market area of facility $n$ for class $m$ customers, and the number of class $m$ customers that will be attracted to facility $n$ becomes

$$Q_{mn} = \int_{\Omega_{mn}} q_m(x,y) d\Omega, \quad m = 1,2,\ldots,M, \quad n = 1,2,\ldots,N$$  \hspace{1cm} (6)

From the flow conservation principle at the peripheral of facility $n$ we have

$$\int_{r_0} f_m \cdot n \; d\Gamma - Q_{mn} = 0, \quad m = 1,2,\ldots,M, \quad n = 1,2,\ldots,N$$  \hspace{1cm} (7)

Within the city with more than one facility, customers’ choice of facility is not only based on their transportation cost to each of the facility, but also on the in-situ costs at the facilities. The definition of in-situ cost varies for different types of facilities, such as the average prices of consumers’ products in shopping centers, parking costs at airports, etc. Customers make their choices of facility as well as the routes taken to maximize their utilities. Therefore, we define a generalized cost function to represent the total cost perceived by a customer of class $m$ to patronize a facility as a function of the relevant attributes of the in-situ costs at the facility and transportation cost to that facility,

$$G_m[H(x,y),O_n] = \sum_{k=1}^{K} w_{mk} p_{mnk} + C_m[H(x,y),O_n] = \bar{u}_{mn} + C_m[H(x,y),O_n]$$  \hspace{1cm} (8)

where $G_m[H(x,y),O_n]$ is the generalized cost for class $m$ customers from their home location $H(x,y)$ to patronize facility $n$ ($n = 1, 2, \ldots, N$). $\bar{u}_{mn} = \sum_{k=1}^{K} w_{mk} p_{mnk}$ represents the total in-situ cost at the facility $n$, where $K$ attribute measures $p_{mnk}$ for $k = 1,\ldots,K$ are considered for the facility $n$, each with an associated weight $w_{mk}$ perceived by class $m$ customers. $C_m[H(x,y),O_n]$ is the minimum transportation cost for class $m$ customers to travel from home location $H(x,y)$ to their chosen facility $n$. In our model, the weights and the attribute measures are exogenous inputs and their values are given, whereas the equilibrium transportation cost are determined from user-equilibrium conditions.

As discussed previously, the customers patronize the facility with the lowest generalized cost in a deterministic manner. The individual class $m$ customer from home location $H(x,y)$ will choose to patronize facility $n$, so that the following equation is satisfied:

$$G_m[H(x,y),O_n] \geq G_m[H(x,y),O_l], \quad \forall l = 1,2,\ldots,N$$  \hspace{1cm} (9)

Furthermore, with the chosen facility, customers will make their routing decision over the continuum city space in a user-optimal manner, so that the total path cost is minimized:

$$C_m[H(x,y),O_n] = \min_p C_{mp}[H(x,y),O_n] \quad \forall p$$  \hspace{1cm} (10)

where $C_{mp}[H(x,y),O_n] = \int_{p} c_m(x,y,F) ds$ is the transportation cost from $H(x,y)$ to $O_n$, via path $p$ in the continuum space.
3. PROBLEM FORMULATION

The problem of a multi-class customer spatial choice equilibrium with elastic demand can be formulated as the following minimization problem.

Minimise $z = \sum_{m=1}^{M} \sum_{n=1}^{N} u_{mn} Q_{mn} + \int_{\Omega} \left( \sum_{m=1}^{M} a_{m} f_{m} + \sum_{m=1}^{M} b_{ml} \frac{1}{2} \left| f_{m} \right| + \sum_{m=1}^{M} D_{m}^{-1}(\xi) d\xi \right) d\Omega$  

subject to

\[ \nabla f_{m} + q_{m} = 0, \forall (x, y) \in \Omega, \forall m = 1, \ldots, M \]  

\[ f_{m} = 0, \forall (x, y) \in \Gamma, \forall m = 1, \ldots, M \]  

\[ Q_{mn} + \int_{\Gamma_{cn}} f_{m} \cdot n d\Gamma = 0, \forall (x, y) \in \Gamma_{cn}, \forall n = 1, \ldots, N, \forall m = 1, \ldots, M \]  

Consider the following Lagrangian function:

\[ \Pi = \int_{\Omega} \sum_{m=1}^{M} a_{m} f_{m} + \sum_{m=1}^{M} b_{ml} \frac{1}{2} \left| f_{m} \right| + \sum_{m=1}^{M} D_{m}^{-1}(\xi) d\xi + \sum_{m=1}^{M} u_{m} (\nabla f_{m} + q_{m}) d\Omega \]

\[ + \int_{\Gamma} \sum_{m=1}^{M} \left( w_{m} \cdot f_{m} + \sum_{m=1}^{M} \nabla Q_{mn} + \sum_{m=1}^{M} \Pi_{mn} \left( Q_{mn} + \int_{\Gamma_{cn}} f_{m} \cdot n d\Gamma \right) \right) d\Gamma \]  

where $u_{m}, w_{m} = \{ w_{mx}, w_{my} \}$, and $\pi_{mn}$ are the Lagrangian multipliers associated with equations (11b), (11c), and (11d), respectively. Applying variational principle to the Lagrangian function, we can show that

\[ \delta \Pi = \int_{\Omega} \sum_{m=1}^{M} a_{m} f_{m} \delta f_{m} + \frac{1}{2} \sum_{m=1}^{M} b_{ml} \left( \frac{\delta f_{m}}{f_{m}} \right) + \sum_{m=1}^{M} D_{m}^{-1}(\xi) \delta \xi + \sum_{m=1}^{M} u_{m} (\nabla f_{m} + q_{m}) \delta \Omega \]

\[ + \int_{\Gamma} \sum_{m=1}^{M} (w_{m} \cdot \delta f_{m} + f_{m} \cdot \delta w_{m}) d\Gamma + \sum_{m=1}^{M} \sum_{m=1}^{N} u_{mn} \delta Q_{mn} + \sum_{m=1}^{M} \sum_{m=1}^{N} \pi_{mn} \delta Q_{mn} - \sum_{m=1}^{M} \sum_{m=1}^{N} \int_{\Gamma_{cn}} \pi_{mn} \delta f_{m} d\Gamma \]  

As $\nabla (u_{m} \delta f_{m}) = \delta f_{m} \nabla u_{m} + u_{m} \nabla \delta f_{m}, \forall m = 1, \ldots, M$, and by the divergence theorem, we have

\[ \sum_{m=1}^{M} \int_{\Omega} \nabla (u_{m} \delta f_{m}) d\Omega = \sum_{m=1}^{M} \int_{\Gamma} u_{m} \delta f_{m} \cdot n d\Gamma + \sum_{m=1}^{M} \sum_{m=1}^{N} \int_{\Gamma_{cn}} u_{m} \delta f_{m} \cdot n d\Gamma \]  

Substituting equation (14) into equation (13), we can show that
\[ \delta \Pi = \sum_{m=1}^{M} \int_{\Omega} \left[ \left( a_m + \sum_{n} b_{mn} f_n \right) \frac{f_m}{f_m^2} - \nabla u_m \right] \cdot \delta f_m + \left[ u_m - D_m^{-1}(q_m) \right] \delta q_m + (\nabla f_m + q_m) \delta u_m \, d\Omega \]
\[ + \sum_{m=1}^{M} \int_{\Gamma} (w_m + u_m \mathbf{n}) \cdot \delta f_m + f_m \cdot \delta w_m \, d\Gamma + \sum_{m=1}^{M} \sum_{n=1}^{N} \int_{\Gamma_{cn}} (u_m + \pi_{mn}) \mathbf{n} \cdot \delta f_m \, d\Gamma \]
\[ + \sum_{m=1}^{M} \sum_{n=1}^{N} (\bar{u}_{mn} + \pi_{mn}) \delta Q_{mn} + \sum_{m=1}^{M} \sum_{n=1}^{N} Q_{mn} \delta u_{mn} + \sum_{m=1}^{M} \sum_{n=1}^{N} \left( Q_{mn} + \int_{\Gamma_{cn}} f_m \cdot \mathbf{n} \, d\Gamma \right) \delta \pi_{mn} \]  

Because \( \delta f_m \) vanishes on \( \Gamma \) and \( \delta \bar{u}_{mn} \) vanishes on \( \Gamma_{cn} \), \( \delta f_m \), \( \delta q_m \), and \( \delta u_m \) are arbitrary functions in \( \Omega \), \( \delta w_m \) is an arbitrary function on \( \Gamma \), and \( \delta f_m \), \( \delta Q_{mn} \), and \( \delta \pi_{mn} \) are arbitrary functions on \( \Gamma_{cn} \), we can easily show that for the stationary point of the Lagrangian \( \delta \Pi = 0 \),

\[ \left( a_m + \sum_{n} b_{mn} f_n \right) \frac{f_m}{f_m^2} - \nabla u_m = 0, \forall (x, y) \in \Omega, m = 1, \ldots, M \]  

(16)
\[ u_m - D_m^{-1}(q_m) = 0, \forall (x, y) \in \Omega, m = 1, \ldots, M \]  

(17)
\[ \nabla f_m + q_m = 0, \forall (x, y) \in \Omega, m = 1, \ldots, M \]  

(18)
\[ f_m = 0, \forall (x, y) \in \Gamma, m = 1, \ldots, M \]  

(19)
\[ Q_{mn} + \int_{\Gamma_{cn}} f_m \cdot \mathbf{n} \, d\Gamma = 0, \forall (x, y) \in \Gamma_{cn}, n = 1, \ldots, N, m = 1, \ldots, M \]  

(20)
\[ \bar{u}_{mn} + \pi_{mn} = 0, \forall (x, y) \in \Omega, n = 1, \ldots, N, m = 1, \ldots, M \]  

(21)
\[ u_m + \pi_{mn} = 0, \forall (x, y) \in \Gamma_{cn}, n = 1, \ldots, N, m = 1, \ldots, M \]  

(22)

From (16), we have \( f_m \parallel \nabla u_m \), where \( \parallel \) means that the vectors are parallel, and

\[ |\nabla u_m| = a_m + \sum_{l=1}^{M} b_{ml} |f_l| = c_m \]  

(23)

Therefore, \( |\nabla u_m| \) can be interpreted as the transportation cost per unit distance for class \( m \) customers. From (21) and (22), we have

\[ u_m = \bar{u}_{mn}, \forall (x, y) \in \Gamma_{cn}, n = 1, \ldots, N, m = 1, \ldots, M \]  

(24)

This implies that the cost potential \( u_m \) at the periphery of facility \( n \) has the same value as the in-situ cost as perceived by class \( m \) customers at that facility. For any used path \( p \) of class \( m \) customers from home location (H) to facility \( n \), the generalized cost is

\[ G_p = \bar{u}_{mn} + \int_p c_m \, ds = \bar{u}_{mc} + \int_p |\nabla u_m| \, ds = \bar{u}_{mn} + \int_p \nabla u_m \cdot ds = u_m(H) \]  

(25)

For any unused path \( \bar{p} \) of class \( m \) customers from H to facility \( n \), the generalized cost is

\[ G_{\bar{p}} = \bar{u}_{mn} + \int_{\bar{p}} c_m \, ds = \bar{u}_{mn} + \int_{\bar{p}} |\nabla u_m| \, ds \geq \bar{u}_{mn} + \int_{\bar{p}} \nabla u_m \cdot ds = u_m(H) \]  

(26)
From the above two equations, the generalized costs of all the used paths for class \( m \) customers are equal, and are less than or equal to those of the unused paths. In the city with more than one facility, we can interpret this as customers choosing the facility for which the total cost for them to travel and use is the minimum. Moreover, for the same chosen facility, customers will choose the path of least transportation cost. Thus, this model ensures that the customers of any class, who have perfect information about the transportation system, make rational decisions in a user-optimal manner (Wardrop, 1952).

4. SOLUTION ALGORITHM

We use the finite element method (Zienkiewicz and Taylor, 1989) to solve the minimization problem (11). The whole city is first discretized into a set of triangular finite elements (Cheung et al., 1996), as shown in Figure 2. Although the continuum city region is divided into smaller sub-regions via the discretization process during the solution stage, the continuum method is still superior than the discrete network approach at least at a global level of modeling, as the size of each element is flexible and each element may already represent thousands of street sections. Moreover, as the traffic demand generated from the housing or other developments is largely continuously distributed within the city, the local variation of demand on a two-dimensional space can be more realistically modeled by piece-wise continuous functions within each element, and the global variation of demand can then be represented by a number of individual elements joining up to form the city region. This contrasts with the discrete modeling approach where the continuous traffic demand has to be modeled by assuming that the demand in a zone is concentrated at a point called centroid and they are assigned to the network through some pre-specified centroid connectors.

The Lagrangian of the minimization problem is now modified as follows:

\[
\Pi = \sum_{e \in \Omega} \sum_{m=1}^{M} \int_{\Omega_e} a_m \| \mathbf{f}_m \| + \sum_{l=1}^{M} b_{ml} \| \mathbf{f}_m \| - \int_{0}^{\Gamma_m} D_m^{-1}(\xi) d\xi + u_m (\nabla \mathbf{f}_m + q_m) d\Omega \\
+ \sum_{e \in \Omega} \sum_{m=1}^{M} \int_{\partial\Omega_e \setminus \Gamma} \mathbf{w}_m \cdot \mathbf{f}_m d\Gamma - \sum_{n=1}^{N} \sum_{m=1}^{M} \int_{\partial\Omega_n \setminus \Gamma_{cn}} \beta_{mn} (\mathbf{f}_m \cdot \mathbf{n}) d\Gamma \\
+ \sum_{n=1}^{N} \sum_{m=1}^{M} \int_{\partial\Omega_n \setminus \Gamma_{cn}} \sigma (u_m - \bar{u}_{mn}) d\Gamma 
\]  

(27)

where \( \Omega_e \) is the sub-domain of an element \( e \), \( a_m \) and \( b_{ml} \), \( l = 1, \ldots, M \), are the coefficients of the cost-flow relationship for class \( m \) customers that are assumed to be constant within the element. \( \mathbf{f}_m \) and \( u_m \) are respectively the flow vector and cost potential for the class \( m \) customers. \( \partial\Omega_e \setminus \Gamma \) and \( \partial\Omega_n \setminus \Gamma_{cn} \) indicate the elements that contain, respectively, the boundary of the facilities and the outer bound of the modeled city. \( \mathbf{w} \) and \( \sigma \) are, respectively, the Lagrange multipliers along \( \Gamma \) and \( \Gamma_{cn} \). The solution of the minimization problem in equation (11) can be found by locating the stationary point of the Lagrangian in equation (27). For the second term of equation (27), because there is no traffic flow along the boundary \( \Gamma \) it can be replaced by a much stronger and convenient expression,
where $N_\Omega$ is the number of nodes along $\Gamma$, $f_{mxi}$ and $f_{myi}$ are, respectively, the flow flux in the $x$ and $y$ directions at node $i$ for class $m$, and $w_{mxi}$ and $w_{myi}$ are the corresponding Lagrange multipliers. The zero boundary flow condition can be guaranteed by forcing $f_{mxi} = f_{myi} = 0$ for all nodes along $\Gamma$. As the third term of that equation (27) represents the sum of the total customer costs of using each of the facilities, it can be replaced by a much more convenient expression,

$$
- \sum_{n=1}^{N} \sum_{i=1}^{S_n} \sum_{m=1}^{M} \bar{u}_{mn} L_i \left( \frac{f_{mj} + f_{mk}}{2} \right) \cdot n_i
$$

(29)

where $S_n$ is the number of line segments on the boundary of facility $n$, $L_i$, and $n_i$ are, respectively, the length and the unit normal vector of the line segment $i$ around the boundary of the facility $n$, and $f_{mj}$, and $f_{mk}$ are, respectively, the flow vector of the initial and end nodes of the line segment $i$. Similarly, as it is proved that the cost potential at the boundary of the facility is equal to the total attribute measure of that facility, the last term of equation (27) can be replaced by the following expression,

$$
\sum_{n=1}^{N} \sum_{i=1}^{N_n} \sum_{m=1}^{M} \sigma_i (u_{mi} - \bar{u}_{mn})
$$

(30)

where $N_n$ is the number of nodes on the boundary of facility $n$, $u_{mi}$ is the cost potential of class $m$ at node $i$, and $\sigma_i$ is its Lagrange multiplier at node $i$. The conditions of equation (30) can be ensured by forcing $u_{mi} - \bar{u}_{mn} = 0$ for all nodes along the boundaries of facilities. With all of the above modifications, equation (27) can be modified as follows,

$$
\Pi = \sum_{e \in \Omega} \sum_{m=1}^{M} \int_{\Omega_e} a_m |f_m| + \sum_{l=1}^{M} \frac{b_{ml}}{2} |f_m| |f_l| - \sum_{m=1}^{M} D_m^{-1}(\xi) \xi^2 + u_m (\nabla f_m + a_m) d\Omega
$$

$$
+ \sum_{i=1}^{N_\Omega} \sum_{m=1}^{M} \left( w_{mxi} f_{mxi} + w_{myi} f_{myi} \right) - \sum_{n=1}^{N} \sum_{i=1}^{S_n} \sum_{m=1}^{M} \bar{u}_{mn} L_i \left( \frac{f_{mj} + f_{mk}}{2} \right) \cdot n_i
$$

$$
+ \sum_{n=1}^{N} \sum_{i=1}^{N_n} \sum_{m=1}^{M} \sigma_i (u_{mi} - \bar{u}_{mn})
$$

(31)

The three-node linear triangular element is used to approximate the functions of the variables over the solution space. The value of the flow vectors and the cost potential within the element are expressed as follows,

$$
f_{mxi}(x,y) = N_i(x,y)f_{mxi} + N_j(x,y)f_{mji} + N_k(x,y)f_{mik}
$$

(32)

$$
f_{myi}(x,y) = N_i(x,y)f_{myi} + N_j(x,y)f_{myj} + N_k(x,y)f_{myk}
$$

(33)

$$
u_{mi}(x,y) = N_i(x,y)u_{mi} + N_j(x,y)u_{mj} + N_k(x,y)u_{mk}
$$

(34)
where the subscripts $i, j,$ and $k$ represent the three nodes of the triangular element, and the variables with such subscripts are their values at the nodal point. $N_i, N_j,$ and $N_k$ are the linear interpolation functions of the element, and are expressed as follows,

$$N_i(x,y) = \frac{1}{2\Delta} (\alpha_i + \beta_j x + \xi_k y)$$ (35)

$$N_j(x,y) = \frac{1}{2\Delta} (\alpha_j + \beta_i x + \xi_k y)$$ (36)

$$N_k(x,y) = \frac{1}{2\Delta} (\alpha_k + \beta_j x + \xi_i y)$$ (37)

The values of $\alpha, \beta, \xi,$ and $\Delta$ are explicitly related to the nodal co-ordinates $(x_i, y_i), (x_j, y_j),$ and $(x_k, y_k)$ at nodes $i, j,$ and $k,$ respectively, of the triangular element. The explicit forms of $\alpha, \beta, \xi,$ and $\Delta$ are

$$\alpha_i = x_j y_k - x_k y_j, \quad \beta_i = y_j - y_k, \quad \xi_i = x_k - x_j,$$

and $\Delta = (\beta_j \xi_k - \beta_k \xi_j)/2.$ The variations of the flow vector and cost potential over the problem domain are expressed as nodal values of the generated triangular elements and the interpolation functions. Substituting Equations (32) to (37) into equation (31) and taking integration, the Lagrangian can now be expressed as a function of all nodal variables, that is $\Pi(\Psi),$ where

$$\Psi = \text{Col}(f, u, w, \sigma)$$

and

$$f = \text{Col}(f_{mi}, f_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_N)$$ (38)

$$u = \text{Col}(w_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_N)$$ (39)

$$w = \text{Col}(w_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,N_O)$$ (40)

$$\sigma = \text{Col}(\sigma_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_C)$$ (41)

where $M_N, N_O,$ and $M_C$ are, respectively, the total number of node, number of node on the city boundary and the number of nodes on the facilities’ boundary. “Col” represents a column vector. To solve the problem, the stationary point of the Lagrangian should be considered, and is found by the Newtonian algorithm as follows. Let $\Psi^0$ be an approximate solution to the problem and by means of Taylor’s expansion around the point $\Psi^0,$ we have

$$\Pi(\Psi) \approx \Pi(\Psi^0) + R^T(\Psi - \Psi^0) + \frac{1}{2} (\Psi - \Psi^0)^T J (\Psi - \Psi^0)$$ (42)

where $R = \nabla \Pi(\Psi^0)$ is the residual vector of the first derivatives evaluate at $\Psi^0,$ and $J = \nabla^2 \Pi(\Psi^0)$ is the Jacobian matrix evaluate at $\Psi^0.$ For the stationary point the derivatives of $\Pi$ with respect to all variables vanish. Therefore,

$$\nabla \Pi = R + J(\Psi - \Psi^0) = 0$$ (43)

After rearranging, a better solution can be obtained by

$$\Psi = \Psi^0 - J^{-1} R$$ (44)

Therefore, an iterative procedure can be derived as follows.
Solution Procedure:
Step 1: Find an initial solution $\Psi^{(0)}$. Set $k = 0$.
Step 2: Evaluate $R(\Psi^{(k)})$ and $J(\Psi^{(k)})$.
Step 3: If $\left| R(\Psi^{(k)}) \right| < \varepsilon$, and an acceptable error, then stop and $\Psi^{(k)}$ is the solution.
Step 4: Otherwise, find $\Psi^{(k+1)} = \Psi^{(k)} - J(\Psi^{(k)})^{-1} R(\Psi^{(k)})$.
Step 5: Replace $\Psi^{(k)}$ by $\Psi^{(k+1)}$. Set $k = k + 1$ and go to Step 2.

The total cost consumed by each class of customers in the city, which is a useful performance measures of the system, can be determined by

$$T_m = \sum_{e \in \Omega} \int_{\Omega_e} u_m D_m(u_m) \, d\Omega, \ m = 1, \ldots, M$$  \hspace{1cm} (45)

For the case of elastic demand, another performance measure known as the social welfare (or net economic benefit) can also be evaluated by

$$S = \sum_{m=1}^2 \sum_{e \in \Omega} \int_{\Omega_e} \int_{\xi_m} u_m D_m(\xi) \, d\xi \, d\Omega$$  \hspace{1cm} (46)

5. NUMERICAL EXAMPLES

5.1 Example 1: Fixed Demand

In this paper, we present two hypothetical examples to demonstrate the methodology. In Example 1, we consider a modeled city with two facilities as shown in Figure 1. The finite element mesh that is used for analysis is shown in Figure 2. There are two classes of customers, each of which has distinct demand and cost functions. Different classes can be used to represent the heterogeneous characteristics of customers, such as the different values of times for low and high income groups or the different mobility of light and heavy vehicles. The travel demands are fixed at 150 and 100 veh/h/km for class 1 and 2, respectively. The total demand (classes 1 and 2 combined) in the system is 147,779 veh/h. The cost-flow functions are given as

$$c_1 = 0.01 + 0.5 \times 10^{-4} |f_1| + 0.4 \times 10^{-4} |f_2|$$  \hspace{1cm} (47)

$$c_2 = 0.05 + 0.4 \times 10^{-4} |f_1| + 0.6 \times 10^{-4} |f_2|$$  \hspace{1cm} (48)

throughout the city, where $c_1$ and $c_2$ are expressed in terms of h/km. The in-situ costs for class 1 customers are 2.0 and 4.0 hours for facilities 1 and 2, respectively, and those for class 2 customers are 2.5 and 3.5 hours. The model can be solved by the Newtonian algorithm with an acceptable error $\varepsilon$ of $10^{-3}$. The results of flow vectors for class 1 and class 2 customers are shown in Figures 3 and 4, respectively. These figures show the paths that are chosen by the customers from their home locations to their chosen facilities. The catchment areas of the two facilities can also be visualized from the figures. The flow intensities and generalized costs of the two customers classes are shown in Figures 5 to 8. Note that the unit for flow intensity is veh/h/km, and that for the generalized cost is hours. The total generalized cost incurred by all

customers is 672,679 veh-h, with 390,111 and 282,568 veh-h for class 1 and class 2 customers, respectively.

The market shares of facility 1 for class 1 and class 2 customers are 70.8% and 47.9%, respectively, whereas those of facility 2 are 29.2% and 52.1%. For class 1 customers, facility 1 takes up a larger proportion of demand as its facility in-situ cost for class 1 customers is much less than that of facility 2, which can compensate for the extra congestion that is incurred by the customers around that facility. However, for class 2 customers, the situation is the reverse. Although the in-situ cost of facility 1 for class 2 customers is less than that of facility 2, more customers choose to use facility 2 because the congestion cost that is incurred by class 1 customers around facility 1 is relatively large, and cannot be compensated for by the lower in-situ cost at facility 1. Consequently, class 2 customers choose to use a more expensive but less congested facility. This result demonstrates how customers of different classes affect each other in their choices of route in the road network and facility.

5.2 Example 2: Elastic Demand

The elastic demand for the two customer classes is also considered. The demand functions for customer classes 1 and 2 are taken as follows:

\[ D_1(u_1) = 400e^{-0.3u_1} \]
\[ D_2(u_2) = 350e^{-0.2u_2} \]

(49) (50)

where \( u_1 \) and \( u_2 \) are, respectively, the generalized cost potentials for class 1 and class 2 customers at their home locations. The model can also be solved by the Newtonian algorithm, which gives similar patterns of flow vectors, flow intensities, and generalized costs as for the case of fixed demand. The distribution patterns of demand for class 1 and class 2 customers are shown in Figures 9 and 10, respectively. Generally, the demand intensity decreases with the distance away from the facilities due to increasing generalized cost. The total demand (classes 1 and 2 combined) is 146,783 veh/h for this elastic demand case. The total generalized cost incurred by all customers is 659,986 veh-h, with 279,544 and 380,442 veh-h for class 1 and class 2 customers, respectively. The social welfare for the transportation system is 626,919 veh-h.

6. CONCLUSIONS

The continuum approximation of network flow by Wong and Yang (1999) and Yang and Wong (2000) for a city with a very dense transportation network and multi-competing facilities has been extended to deal with the interactions of multi-class users. In this study, each class of users is assumed to have its own transportation cost function, demand function, and facility in-situ costs. Each user class affects the others through congestion externality in the transportation system. A mathematical program for this multi-class problem has been formulated, and the user equilibrium conditions have been satisfied. The finite element method has been used to discretize the problem, and a Newtonian algorithm has been developed to determine the solution. Both cases of fixed and elastic demand have been used as numerical examples to demonstrate the effectiveness of the proposed methodology. There are two directions for further research. The first is to develop a multi-class model with asymmetric cost functions. The second is to consider the elastic market externality function to account for the congestion effect at the facilities.
Figure 1: The Modeled City

Figure 2: The Finite Element Mesh

Figure 3: Flow Pattern of Class 1 Customers

Figure 4: Flow Pattern of Class 2 Customers

Figure 5: Flow Intensity of Class 1 Customers

Figure 6: Flow Intensity of Class 2 Customers

Figure 7: Total Cost for Class 1 Customers

Figure 8: Total Cost for Class 2 Customers

Figure 9: Demand of Class 1 Customers (Elastic Demand Case)

Figure 10: Demand of Class 2 Customers (Elastic Demand Case)
ACKNOWLEDGEMENTS

We gratefully acknowledge the support that was given to this project by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No.: HKU7015/00E), and an Outstanding Young Researcher Award 2000 from the University of Hong Kong.

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